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From Quantum Many Body Systems to Nonlinear Schrödinger Equations

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From Quantum Many Body Systems to Nonlinear Schrödinger Equations

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Dedicated to

my beloved mother and father

and to my beloved wife Danyang.

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From Quantum Many Body Systems to Nonlinear Schrödinger Equations

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The derivation of nonlinear dispersive PDE, such as the nonlinear Schrödinger (NLS) or nonlinear Hartree equations, from many body quantum dynamics is a central topic in mathematical physics, which has been approached by many authors in a variety of ways. In particular, one way to derive NLS is via the Gross-Pitaevskii (GP) hierarchy, which is an infinite system of coupled linear non-homogeneous PDE. In this thesis we present two types of results related to obtaining NLS via the GP hierarchy. In the first part of the thesis, we derive a NLS with a linear combination of power type nonlinearities in $\mathbb{R}^d$ for $d = 1, 2$. In the second part of the thesis, we focus on considering solutions to the cubic GP hierarchy and we prove unconditional uniqueness of low regularity solutions to the cubic GP hierarchy in $\mathbb{R}^d$ with $d \geq 1$: the regularity of solution in our result coincides with known regularity of solutions to the cubic NLS for which unconditional uniqueness holds.

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Chapter 1

Introduction

1.1 Background review

The nonlinear Schrödinger equation (NLS) is a macroscopic model for a quantum mechanical system, with different type of nonlinearities depending on the way one models the interaction potential (cubic, quintic, Hartree, etc.) in a quantum many body system. The research efforts aimed at providing a rigorous derivation of nonlinear dispersive equations as mean field limits of N-body Schrödinger dynamics have a long and rich history as well as feracious recent activities.

The first results on the derivation of nonlinear Hartree equations (NLH) were due to Hepp [28], and Ginibre and Velo [20, 21]. Their techniques are based on embedding the N-body Schrödinger equation into the second quantized Fock-space representation. In [35, 36], Lanford had employed the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy to study N-body systems in classical mechanics in the limit $N \to \infty$. In early 80's Spohn [49] derived a NLS via employing the BBGKY hierarchy. More recently, Erdös, Schlein and Yau further developed the BBGKY approach, and gave the first derivation of cubic NLS in 3D in their celebrated works [14, 15, 16, 17]. They proved that under pairwise interaction model, the k-particle density matrix for BBGKY hierarchy converges to that of the infinite hierarchy (Gross-Pitaevskii hierarchy), which is actually governed by the solution of the cubic nonlinear Schrödinger equation. We will provide a brief description of their approach in section §1.1.1.

Let us also mention that recently in [45], Rodnianski and Schlein proved estimates on the convergence rate of the evolution in the mean field limit using the Fock space approach. Their results were extended with second-order corrections in the two-body interaction setting by Grillakis, Machedon and Margetis [24, 25], and three-body interaction setting by X. Chen [10].

1.1.1 From quantum many-body system to NLS

In an N-particle bosonic system in $\mathbb{R}^3$, the time evolution of the wave function $\psi_N(t)$ (which is assumed to have permutation symmetry in all space variables: $\psi_N(t, x_{\sigma(1)}, x_{\sigma(2)} ..., x_{\sigma(N)}) = \psi_N(t, x_1, x_2 ..., x_N)$, for any permutation $\sigma \in S_N$) follows the Schrödinger equation

$$i\partial_t \psi_N(t) = H_N \psi_N(t), \tag{1.1.1}$$

where the Hamiltonian is given by

$$H_N := -\sum_{j=1}^{N} \Delta_j + \sum_{i<j} V_N(x_i - x_j). \tag{1.1.2}$$

Here, $V_N(x) := N^{3\beta} V(N^\beta x)$ with $\beta \in (0, 1)$ (we remark that the case $\beta = 1$ is much more difficult to control [14, 15, 16, 17]). The pair interaction potential V is assumed to be rotationally symmetric, and it satisfies certain regularity properties. Since the number of particles N is usually large (up to $\sim 10^{30}$ in the case of boson stars) and particles interact with each other, it is difficult to find an explicit solution to (1.1.1). The numerical methods cannot help either because of the size of N. Meanwhile, in the spirit of statistical mechanics the observable properties reflected by averaging over individual particles are more interesting. Though N is finite, one expects that the limit $N \to \infty$ is a good approximation of the system (for further explanations, see §4.1 of [48]). In fact, the solution is governed by a nonlinear Schrödinger equation in a macroscopic view, and this is rigorously proved by Erdös-Schlein-Yau [14, 15, 16, 17].

In order to consider such a limit as $N \to \infty$, one defines the *N-particle density matrix* $\gamma_N \in \mathbb{C}(\mathbb{R}^{dN} \times \mathbb{R}^{dN})$ as

$$\gamma_N(t, \mathbf{x}_N, \mathbf{x}'_N) := \psi_N(t, \mathbf{x}_N)\overline{\psi}_N(t, \mathbf{x}'_N), \tag{1.1.3}$$

where $\overline{\psi}_N$ denotes the complex conjugate of ψ_N.

Note that the L^2-normalization of ψ_N implies that $\mathrm{Tr}\gamma_N = 1$. For any integer k, by tracing out the last $N-k$ space variables, we obtain the *k-particle marginal density matrix* $\gamma_N^{(k)}$ as

$$\gamma_N^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) := \int_{\mathbb{R}^{d(N-k)}} \psi_N(t, \mathbf{x}_k, \mathbf{x}_{N-k})\overline{\psi}_N(t, \mathbf{x}'_k, \mathbf{x}_{N-k}) d\mathbf{x}_{N-k}, \tag{1.1.4}$$

where $\mathbf{x}_k = (x_1, x_2, \cdots, x_k)$ and $\mathbf{x}'_k = (x'_1, x'_2, \cdots, x'_k)$ are both in $\mathbb{R}^{dk}$, and $\mathbf{x}_{N-k} = (x_{k+1}, \cdots, x_N) \in \mathbb{R}^{d(N-k)}$.

Then it can be shown that the marginal density matrices $\gamma_N^{(k)}$ satisfy the following coupled system of equations. This set of equations is known as the *BBGKY hierarchy* .

$$\begin{aligned}
& i\partial_t \gamma_N^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) \\
&= \sum_{j=1}^{k} (-\Delta_{x_j} + \Delta_{x'_j}) \gamma_N^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) \\
&\quad + \frac{1}{N} \sum_{1 \le j < l \le k} \big(V_N(x_j - x_l) - V_N(x'_j - x'_l) \big) \gamma_N^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) \\
&\quad + \frac{N-k}{N} \sum_{j=1}^{k} \int dx_{k+1} \big(V_N(x_j - x_{k+1}) \\
&\qquad\qquad - V_N(x'_j - x_{k+1}) \big) \gamma_N^{(k+1)}(t, \mathbf{x}_k, x_{k+1}, \mathbf{x}'_k, x_{k+1})
\end{aligned} \tag{1.1.5}$$

Formally take the limit $N \to \infty$ in (1.1.5) to observe that, V_N converges weakly to $(\int V(x)dx)\delta$, where δ denotes the delta distribution. The BBGKY hierarchy (1.1.5) converges to the following (cubic) *Gross-Pitaevskii (GP) hierarchy* of equations:

$$\begin{aligned}
& i\partial_t \gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) \\
&= \sum_{j=1}^{k} (-\Delta_{x_j} + \Delta_{x'_j}) \gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) \\
&\quad + \sum_{j=1}^{k} b_0 \int dx_{k+1} \big(\delta(x_j - x_{k+1}) - \delta(x'_j - x_{k+1}) \big) \gamma^{(k+1)}(t, \mathbf{x}_k, x_{k+1}, \mathbf{x}'_k, x'_{k+1})
\end{aligned} \tag{1.1.6}$$

where $b_0 = \int V(x)dx$ is the coupling constant. This heuristic derivation can be justified rigorously, and that has been done in [14, 15, 16, 17].

The cubic GP hierarchy can also be written in the following way:

$$i\partial_t \gamma^{(k)} = (-\Delta_{\mathbf{x}_k} + \Delta_{\mathbf{x}'_k})\gamma^{(k)} + \lambda B_{k+1}\gamma^{(k+1)}, \quad \forall k \in \mathbb{N}, \tag{1.1.7}$$

with equations in (1.1.7) coupled by the *contraction operator* B_{k+1},

$$B_{k+1} = \sum_{j=1}^{k} B_{j;k+1} = \sum_{j=1}^{k} (B^+_{j;k+1} - B^-_{j;k+1}),$$

where each $B^+_{j;k+1}$ contracts the triple x_j, x_{k+1}, x'_{k+1}:

$$\begin{aligned} &\left(B^+_{j;k+1}\gamma^{(k+1)}\right)(t, \mathbf{x}_k, \mathbf{x}'_k) \\ &= \int dx_{k+1} dx'_{k+1} \delta(x_j - x_{k+1})\delta(x_j - x'_{k+1})\gamma^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}'_{k+1}) \\ &= \gamma^{(k+1)}(t, \mathbf{x}_k, x_j, \mathbf{x}'_k, x_j) \end{aligned} \tag{1.1.8}$$

and each $B^-_{j;k+1}$ contracts the triple x'_j, x_{k+1}, x'_{k+1},

$$\begin{aligned} &\left(B^-_{j;k+1}\gamma^{(k+1)}\right)(t, \mathbf{x}_k, \mathbf{x}'_k) \\ &= \int dx_{k+1} dx'_{k+1} \delta(x'_j - x_{k+1})\delta(x'_j - x'_{k+1})\gamma^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}'_{k+1}) \\ &= \gamma^{(k+1)}(t, \mathbf{x}_k, x'_j, \mathbf{x}'_k, x'_j). \end{aligned} \tag{1.1.9}$$

This infinite hierarchy of equations is a good model for the Bose-Einstein condensate. For the mathematical study of Bose-Einstein condensation (BEC) in systems of interacting bosons in the stationary case, we refer to the fundamental works [39, 42, 41, 40] and the references therein.

In the special case of factorized initial data for the GP hierarchy: $\gamma_0^{(k)}(\mathbf{x}_k, \mathbf{x}'_k) = \prod_{j=1}^{k} \phi_0(x_j)\overline{\phi}_0(x'_j)$, the state of a Bose-Einstein condensate can be simply described by the cubic nonlinear Schrödinger equation (NLS). Indeed, in this case, the cubic GP hierarchy admits a solution

$$\gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) = \prod_{j=1}^{k} \phi(t, x_j)\overline{\phi}(t, x'_j),$$

preserving the factorization property as time evolves, if $\phi(t, \cdot)$ solves the cubic NLS

$$i\partial_t \phi(t, \cdot) = -\Delta\phi(t, \cdot) + \lambda|\phi(t, \cdot)|^2\phi(t, \cdot), \;\; \phi(t, \cdot)|_{t=0} = \phi_0. \tag{1.1.10}$$

In this way, the cubic NLS is derived as a dynamical mean field limit of the many body dynamics of an interacting Bose gas, provided that given initial data, a solution to the GP hierarchy is unique. The derivation of NLS described above was for the first time justified by Erdös-Schlein-Yau in [14, 15, 16, 17].

Roughly speaking, Erdös-Schlein-Yau's strategy comprises the following three main parts:

(i) show that each solution of the N-body BBGKY hierarchy $\gamma_N^{(k)}$ admits at least one limit point $\gamma^{(k)}$ as $N \to \infty$;

(ii) prove that any limit point $\gamma^{(k)}$ satisfies the GP hierarchy;

(iii) establish uniqueness of solutions to the GP hierarchy. In particular, it is proved that for factorized initial data, the solutions to the GP hierarchy are determined by a cubic NLS.

In this program, the proof of the uniqueness theorem part (iii) is very involved, one of the difficulties being the factorial growth of the number of terms from iterated Duhamel expansions. The authors give a sophisticated combinatorial argument that settled this problem by a clever re-grouping of Feynman graph expansions.

Later in [34], Klainerman and Machedon found a shorter proof of uniqueness of solutions to the 3D cubic GP hierarchy in a different solution space, provided that solutions obey a priori bound,

$$\int_0^T \|R^{(k)} B_{j;k+1}\gamma^{(k+1)}(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} dt < C^k, \quad \forall k \in \mathbb{N}, \tag{1.1.11}$$

where $R_j = (-\Delta_{x_j})^{1/2}$, $R'_j = (-\Delta_{x'_j})^{1/2}$ and $R^{(k)} = \prod_{j=1}^k R_j \prod_{j=1}^k R'_j$. The approach is in part motivated by the authors' previous work on the space-time estimates [33]. In [34], Klainerman and Machedon gave a concise reformulation of the Erdös-Schlein-Yau combinatorial method [14, 15, 16, 17], and presented it as an elegant board game argument. By taking advantage of the space-time estimate obtained from free evolving Schrödinger equations, they achieve a comparatively simpler analysis on the contributions of expansion terms. The uniqueness theorem of [34] is *conditional* due to the hypothesis (1.1.11). Subsequent works like [32] by Kirkpatrick, Schlein and Staffilani, and [7] by Chen and Pavlović proceed along their lines when considering the Bose gas with pair and three-body interactions respectively, and the solutions obtained in both [32] and [7] are shown to satisfy the relevant condition of type (1.1.11).

The approach of Klainerman and Machedon was used in various recent works for the derivation of the NLS from interacting Bose gases [8, 11, 12, 9]. The method also inspired the analysis of the Cauchy problem for the GP hierarchy, which was initiated in [6] by T. Chen and N. Pavlović and continued by P. Gressman, V. Sohinger and G. Staffilani in [23], T. Chen and K. Taliaferro in [9], etc.

1.2 Our contribution

Our contribution presented in this thesis consists of two parts.

- Generalizing the BBGKY approach, we derive a nonlinear Schrödinger equation with linear combination of power type nonlinearities in 1D and 2D.

- In a joint work with Y. Hong and K. Taliaferro, we prove unconditional uniqueness of low regularity solutions to the cubic GP hierarchy in $\mathbb{R}^d$: the regularity of solution in our result coincides with known regularity of solutions to the cubic NLS for which unconditional uniqueness is known.

1.2.1 Derivation of a NLS with a general power-type nonlinearity

The work of deriving a NLS with a linear combination of power type nonlinearities is motived by the work of K. Kirkpatrick, B. Schlein and G. Staffilani [32] and the work of T. Chen and N. Pavlović [7]. The authors consider a quantum model with 2-body interactions [32] and 3-body interactions [7] respectively and obtain cubic and quintic NLS correspondingly that correctly describes the system.

Above mentioned works prior to [7] considered many body systems with pair interactions only. However, for certain situations, more general interactions are important in the sense that they provide a more accurate model for the system dynamics. This is the case, for instance, when the Bose gas interacts with the background field of matter (such a photons). Then averaging over the latter typically leads to a model with a linear combination of n-body interactions, $n = 2, 3, \cdots$.

In [7], Chen and Pavlović predicted that, if both 2-body and 3-body interactions are present in a quantum model, then that would lead (via Gross-Pitaevskii limit) to a NLS with a linear combination of cubic and quintic nonlinearities. We will give a proof of that claim in chapter 3. Actually, we generalize the prediction from [7] and derive the NLS with a finite linear combination of power nonlinearities. We also note that a particular example of such kind of NLS was studied by Tao-Visan-Zhang in [54], in which local and global wellposedness and related questions are explored.

More precisely, in chapter 3, we consider a quantum dynamical system with a finite linear combination of n-body interactions and obtain that the k-particle marginal density of the BBGKY hierarchy converges when particle number goes to infinity. Moreover, the limit solves a corresponding infinite Gross-Pitaevskii hierarchy. The convergence is established by adapting the arguments developed in [14, 15, 16, 17, 32, 7]. For the uniqueness part, we expand the Klainerman-Machedon formulation [34] of the Erdös-Schlein-Yau combinatorial arguments by introducing a different board game to handle the factorial growth in the number of Duhamel terms. The space time bound assumption in [34] is shown to be satisfied.

1.2.2 Low regularity uniqueness of the cubic GP hierarchy

In a recent work [5], T. Chen, Hainzl, Pavlović and Seiringer introduced a new method to prove the 3D unconditional uniqueness of solutions to the cubic GP hierarchy in the space $L^\infty_{t\in[0,T)}\mathfrak{H}^1$ *. Their proof is based on the quantum de Finetti theorem (which is a quantum analogue of the Hewitt-Savage theorem in probability theory) and is significantly simpler compared to the works [14, 15, 16, 17].

For the factorized case, the $\mathfrak{H}^1$ norm of the marginal density $\gamma^{(k)}$ is related to the H^1 Sobolev norm of the solution ϕ to the corresponding NLS. Since the cubic NLS is ill-posed in H^s for $s < s_c := \frac{d}{2} - 1$. In 3D case $s_c = \frac{1}{2}$, so the best regularity space we can hope for unconditional uniqueness happens is $L^\infty_t \mathfrak{H}^s$ for any $s > \frac{1}{2}$, if not the critical case $s = \frac{1}{2}$. The gap between the esult in [5] and this thought motivated us to investigate a possible improvement.

On the other hand, the unconditional uniqueness is known for solutions to the cubic NLS in space H^s with $s > \frac{d}{6}$ if $d = 1, 2$ and $s > s_c = \frac{d-2}{2}$ if $d \geq 3$

* For $\alpha \geq 0$, the space $\mathfrak{H}^\alpha := \big\{\{\gamma^{(k)}\}_{k=1}^\infty \big| \mathrm{Tr}\,(|S^{(k,\alpha)}\gamma^{(k)}|) < M^{2k}, M > 0 \text{ is a constant}\big\}$, where the differential operator $S^{(k,\alpha)} := \prod_{j=1}^k (1-\Delta_{x_j})^{\frac{\alpha}{2}}(1-\Delta_{x_j'})^{\frac{\alpha}{2}}$. In factorized case, $\gamma^{(k)}(\mathbf{x}_k, \mathbf{x}_k') = \prod_{j=1}^k \phi(x_j)\overline{\phi}(x_j')$, so $\gamma^{(k)} \in \mathfrak{H}^\alpha$ if and only if $\|\phi\|_{H^\alpha} < M$.

(see [19, 27]), the endpoint case $s = \frac{1}{6}$ also holds for $d = 1$. It is reasonable to expect to have the unconditional uniqueness for the GP under the same regularity level (same range for s). Together with Y. Hong and K. Taliaferro, we establish the unconditional uniqueness of solutions to the cubic Gross-Pitaevskii hierarchy on $\mathbb{R}^d$ in a low regularity Sobolev space $L^\infty \mathfrak{H}^s$. This is exactly what we proved in chapter 4. In such a way, we extend the recent original work of Chen-Hainzl-Pavlović-Seiringer [5] to lower regularity spaces in all dimensions.

1.3 Notations and basic tools

For the reader's convenience, we record the basic tools and notations that have been used globally throughout this thesis.

1.3.1 Notations

The list of notations are given below with a simple explanation in the attempt to provide a quick indexing. More detailed definitions or explanations of these notations are given in the context that it first appears.

- γ_N: The N-particle density:

$$\gamma_N(t, \mathbf{x}_N, \mathbf{x}'_N) := \psi_N(t, \mathbf{x}_N)\overline{\psi}_N(t, \mathbf{x}'_N).$$

- $\gamma_N^{(k)}$ or $\gamma^{(k)}$: The k-particle marginal density ($\gamma^{(k)}$ is the limit of $\gamma_N^{(k)}$):

$$\gamma_N^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) := \int_{\mathbb{R}^{d(N-k)}} \psi_N(t, \mathbf{x}_k, \mathbf{x}_{N-k})\overline{\psi}_N(t, \mathbf{x}'_k, \mathbf{x}_{N-k}) d\mathbf{x}_{N-k}.$$

- B_{k+1}: The contraction operator B_{k+1} for the cubic GP hierarchy is a sum of k operators $B_{k+1} := \sum_{j=1}^{k} B_{j;k+1}$ where

$$\begin{aligned}
&\left(B_{j;k+1}\gamma^{(k+1)}\right)(t, \mathbf{x}_k, \mathbf{x}'_k) \\
&= \int dx_{k+1}dx'_{k+1}(\delta(x_j - x_{k+1})\delta(x_j - x'_{k+1}) \\
&\qquad - \delta(x'_j - x_{k+1})\delta(x'_j - x'_{k+1}))\gamma^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}'_{k+1}) \\
&= \gamma^{(k+1)}(t, \mathbf{x}_k, x_j, \mathbf{x}'_k, x_j) - \gamma^{(k+1)}(t, \mathbf{x}_k, x'_j, \mathbf{x}'_k, x'_j).
\end{aligned}$$

  The contraction operator for the p-GP hierarchy contracts more positions similarly, see (3.1.8).

- $J^k(\underline{t}_n; \mu)$: The integrand after n-times of Duhamel's expansion in the cubic case:

$$\begin{aligned}
J^k(\underline{t}_n; \mu) := U^{(k)}(t - t_1)B_{\mu(k+1);k+1}U^{(k+1)}(t_1 - t_2)\cdots \\
\cdots U^{(k+n-1)}(t_{n-1} - t_n)B_{\mu(k+n);k+n}\gamma^{(k+n)}(t_n).
\end{aligned}$$

  For general case, see (3.7.2).

- $\mathfrak{H}^\alpha$: The space that the marginal density matrix lives on:

$$\mathfrak{H}^\alpha := \left\{\{\gamma^{(k)}\}_{k=1}^{\infty}\middle|\mathrm{Tr}\ (|S^{(k,\alpha)}\gamma^{(k)}|) < M^{2k}, M > 0 \text{ is a constant}\right\}.$$

- $\mathcal{M}_{k,n}$: The set of all maps μ from $\{k+1, k+2, \cdots, k+n\}$ to $\{1, 2, \cdots, k+n-2\}$ such that $\mu(j) < j$.

- μ_s: A map in $\mathcal{M}_{k,n}$ that is associated to a special upper echelon matrix.

- $< f, g >$: The pairing is defined as $< f, g > := \int f\overline{g}$.
- Tr: The trace operator $\text{Tr}\gamma^{(k)} := \int \gamma^{(k)}(\mathbf{x}_k, \mathbf{x}'_k)|_{\mathbf{x}_k=\mathbf{x}'_k} d\mathbf{x}_k$.
- Tr_{k+1}: The partial trace operator

$$\text{Tr}_{k+1}\gamma^{(k+1)} := \int \gamma^{(k+1)}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x'_{k+1})|_{x_{k+1}=x'_{k+1}} dx_{k+1}.$$

- $|\phi\rangle\langle\phi|^{\otimes k}$: The Dirac notation $|\phi\rangle\langle\phi|^{\otimes k} := \prod_{j=1}^{k} \phi(x_j)\overline{\phi}(x'_j)$.
- $S^{(k,\alpha)}$: The differential operator $S^{(k,\alpha)} := \prod_{j=1}^{k}(1-\Delta_{x_j})^{\frac{\alpha}{2}}(1-\Delta_{x'_j})^{\frac{\alpha}{2}}$.
- $< x >$: The bracket $< x > := (1+|x|^2)^{\frac{1}{2}}, \forall x \in \mathbb{R}^d$.
- $\|f\|_{W^{m,p}}$: The Sobolev norm $\|f\|_{W^{m,p}} := \||\nabla|^m f\|_{L^p}$ where ∇ is the derivative operator. For the special case $p = 2$, we usually use the notation $\|f\|_{H^m} := \|f\|_{W^{m,2}}$.
- $\lesssim_d$: The notation $\lesssim_d$ means less or equal upto a constant which depends on d, i.e., $X \lesssim_d Y \Longleftrightarrow X \leq C_d Y$.
- $U^{(k)}(t)$: The free propagator $U^{(k)}(t) := e^{it(\Delta_{\mathbf{x}_k} - \Delta_{\mathbf{x}'_k})}$.
- $U^{(k)}_{j,j'}$: The free propagator $U^{(k)}_{j,j'} := U^{(k)}(t_j - t_{j'})$.

1.3.2 Basics

Duhamel's formula expresses the solution to a general inhomogeneous linear equation as a superposition of free solutions arising from both the initial data and the forcing term.

Lemma 1.1 (Duhamel's formula). *The solution to the inhomogeneous initial value problem*

$$\partial_t u - Lu = F,$$

for some spatial operator L, is given by

$$u(t) = e^{tL}u(0) + \int_0^t e^{(t-s)L}F(s)ds.$$

Lemma 1.2 (Dispersive estimates). *For all $1 \leq p \leq 2$, we have the following inequalities:*

$$\|e^{it\Delta}u\|_{L_x^{p'}(\mathbb{R}^d)} \lesssim_d |t|^{-d(\frac{1}{p}-\frac{1}{2})}\|u\|_{L_x^p(\mathbb{R}^d)} \tag{1.3.1}$$

and for all $m \in \mathbb{R}$,

$$\|e^{it\Delta}u\|_{W_x^{m,p'}(\mathbb{R}^d)} \lesssim_d |t|^{-d(\frac{1}{p}-\frac{1}{2})}\|u\|_{W_x^{m,p}(\mathbb{R}^d)}, \tag{1.3.2}$$

where p' denotes the Hölder conjugate of p, i.e, $\frac{1}{p} + \frac{1}{p'} = 1$.

The dispersive estimates can be obtained by an interpolation using L^2 conservation law

$$\|e^{it\Delta}u\|_{L_x^2(\mathbb{R}^d)} = \|u\|_{L_x^2(\mathbb{R}^d)} \tag{1.3.3}$$

and the *dispersive inequality*

$$\|e^{it\Delta}u\|_{L_t^\infty} \lesssim_d t^{-\frac{d}{2}}\|u\|_{L_x^1(\mathbb{R}^d)}. \tag{1.3.4}$$

The L^2 conservation is a result of the unitary property of $e^{it\Delta}$ and (1.3.4) is immediate from the fundamental solution of the Schrödinger equation.

By combing Lemma 1.2 with Sobolev inequalities and duality arguments, one can obtain the full Strichartz estimates. Those estimates are are extremely useful when treating with dispersive equations. We record the homogeneous version for Schrödinger equation below.

Lemma 1.3 (Strichartz estimates for Schrödinger)**.** *We call a pair of exponents (q,r) Schrödinger admissible if $2 \leq q,r \leq \infty$, $\frac{2}{q}+\frac{d}{r}=\frac{d}{2}$ and $(q,r,d) \neq (2,\infty,2)$. Then for any admissible exponents (q,r) and $\tilde{q},\tilde{r}$ we have the homogeneous Strichartz estimate*

$$\|e^{it\Delta/2}u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{d,q,r} \|u\|_{L_x^2(\mathbb{R}^d)}. \tag{1.3.5}$$

the dual homogeneous Strichartz estimate

$$\|\int_{\mathbb{R}} e^{-is\Delta/2}F(s)ds\|_{L_x^2}(\mathbb{R}^d) \lesssim_{d,\tilde{q},\tilde{r}} \|F\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}}(\mathbb{R}\times\mathbb{R}^d) \tag{1.3.6}$$

and the retarded Strichartz estimate

$$\|\int_{t'<t} e^{-i(t-t')\Delta/2}F(t')ds\|_{L_t^q L_x^r}(\mathbb{R}^d) \lesssim_{d,q,r,\tilde{q},\tilde{r}} \|F\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}}(\mathbb{R}\times\mathbb{R}^d) \tag{1.3.7}$$

The proof of this theorem in the non-endpoint case (when $q,\tilde{q}\neq 2$) can be found in the original work of Strichartz [52], Ginibre and Velo [22], Yajima [56], Tao [53]. There are also endpoint case Strichartz estimates in general settings covered in [31] by Keel and Tao.

Chapter 2

Erdös-Schlein-Yau combinatorial method in board game form

For both of our works in chapters 3 and 4, we make use of the Erdös-Schlein-Yau [14, 15, 16, 17] combinatorial method presented in the board game form of Klainerman-Machedon [34]. In chapter 3 we generalize this method and in chapter 4 we apply it directly. For the reader's convenience, we summarize the main idea of the combinatorial method in board game form as it was presented in [34].

The generalized version of this combinatorial argument is developed in §3.7. In this chapter, we will recall the notation and main ideas from [34], but proofs of relevant statements are omitted since they are just special cases of those in §3.7 (corresponding to $p_0 = 1$).

Since this combinatorial method is used to prove uniqueness of a hierarchy of linear equations, it suffices to consider the solution of the hierarchy corresponding to the zero initial data. Then one can express the solution $\gamma^{(k)}$ of (1.1.7) with initial data zero in terms of the subsequent terms $\gamma^{(k+1)}$, $\gamma^{(k+2)}$, $\cdots$, $\gamma^{(k+n)}$ by iterating Duhamel's formula. In such a way, one obtains:

$\gamma^{(k)}(t_k, \cdot)$

$$
\begin{aligned}
&= \int_0^{t_k} e^{i(t_k - t_{k+1})\Delta_\pm^{(k)}} B_{k+1}\big(\gamma^{(k+1)}(t_{k+1})\big) dt_{k+1} \\
&= \int_0^{t_k} \int_0^{t_{k+1}} e^{i(t_k - t_{k+1})\Delta_\pm^{(k)}} B_{k+1} e^{i(t_{k+1} - t_{k+2})\Delta_\pm^{(k+1)}} B_{k+2}\big(\gamma^{(k+2)}(t_{k+2})\big) dt_{k+1} dt_{k+2} \\
&= \cdots \\
&= \int_0^{t_k} \cdots \int_0^{t_{k+n-1}} J^k(\underline{t}_{k+n}) dt_{k+1} \cdots dt_{k+n} \qquad (2.0.1)
\end{aligned}
$$

where

$$\Delta_\pm^{(k)} = \Delta_{\mathbf{x}_k} - \Delta_{\mathbf{x}'_k},$$

$$\underline{t}_{k+n} = (t_k, t_{k+1}, \cdots, t_{k+n}),$$

$$J^k(\underline{t}_{k+n}) = e^{i(t_k - t_{k+1})\Delta_\pm^{(k)}} B_{k+1} \cdots e^{i(t_{k+n-1} - t_{k+n})\Delta_\pm^{(k+n-1)}} B_{k+n}(\gamma^{(k+n)}(t_{k+n})).$$

We remark that the essential obstacle in applying the above iteration is that the number of terms in $J^k(\underline{t}_n)$ is very large. More precisely, since each B_{k+i} is a sum of $(k+i-1)$ terms, in the expansion of $J^k(\underline{t}_n)$, there are a total of $k(k+1)\cdots(k+n-1) = \mathcal{O}(n!)$ terms for fixed k. This is exactly the place that board game arguments helps. The key idea of the *board game* argument is that by grouping the large number of integral terms into equivalence classes, whose number is exponential, we can avoid estimating the rapidly increasing number of terms one by one. In order to do that, each equivalence class can be analyzed using Strichartz type estimates for the GP, as it was noticed in [34], and generalized in [7] and in chapter 3 of this thesis.

First we recall how Klainerman-Machedon [34] introduced a mapping that helped them re-express the integrals (2.0.1) via certain matrices. More precisely, let μ be a map from $\{k+1, k+2, \cdots, k+n\}$ to $\{1, 2, \cdots, k+n-1\}$

such that $\mu(2) = 1$ and $\mu(j) < j$ for all j. Let us denote by $\mathcal{M}_{k,n}$ the set of all such maps.

We express J^k in terms of maps μ as follows:

$$J^k(\underline{t}_n) = \sum_{\mu \in \mathcal{M}_{k,n}} J^k(\underline{t}_n; \mu), \tag{2.0.2}$$

where

$$\begin{aligned} J^k(\underline{t}_n; \mu) = U^{(k)}(t - t_1) B_{\mu(k+1);k+1} U^{(k+1)}(t_1 - t_2) \cdots \\ \cdots U^{(k+n-1)}(t_{n-1} - t_n) B_{\mu(k+n);k+n} \gamma^{(k+n)}(t_n). \end{aligned}$$

2.1 Representation of integrals

By the definition of μ, we can represent μ by highlighting exactly one nonzero entry $B_{\mu(k+l),k+l}$ (l-th column, $\mu(k+l)$-th row) in each column of a $(k+n-1) \times n$ matrix. Since $\mu(k+l) < k+l$, we set the remaining entries of the matrix equal to 0. Hence we can visualize a μ via the following matrix:

$$\begin{pmatrix} \mathbf{B_{1;k+1}} & B_{1;k+2} & \cdots & \mathbf{B_{1;k+n}} \\ B_{2;k+1} & B_{2;k+2} & \cdots & B_{2;k+n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{k;k+1} & \mathbf{B_{k;k+2}} & \cdots & B_{k;k+n} \\ 0 & B_{k+1;k+2} & \cdots & B_{k+1;k+n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{k+n-1;k+n} \end{pmatrix} \tag{2.1.1}$$

Henceforth, we can rewrite (2.0.1) as

$$\gamma^{(k)}(t) = \int_0^t \cdots \int_0^{t_n} \sum_{\mu \in \mathcal{M}_{k,n}} J^k(\underline{t}_{k+n}; \mu) dt_1 \ldots dt_n. \tag{2.1.2}$$

Here the time domain $\{t_n \leq t_{n-1} \leq \cdots \leq t\} \subset [0,t]^n$ is the same for all $\mu \in \mathcal{M}_{k,n}$. We now consider the terms $I(\mu,\sigma)$ in the sum $\gamma^{(k)}(t) = \sum_{\mu \in \mathcal{M}_{k,n}} I(\mu,\sigma)$. We have

$$I(\mu,\sigma) = \int_{t_{\sigma(n)} \leq t_{\sigma(n-1)} \leq \cdots \leq t} J^k(\underline{t}_{k+n};\mu) dt_1 \ldots dt_n, \tag{2.1.3}$$

where σ is a permutation of $1,2,\ldots,n$. We associate an integral $I(\mu,\sigma)$ with the following $(k+n)\times n$ matrix:

$$\begin{pmatrix} t_{\sigma^{-1}(1)} & t_{\sigma^{-1}(2)} & \cdots & t_{\sigma^{-1}(n)} \\ \mathbf{B_{1;k+1}} & B_{1;k+2} & \cdots & \mathbf{B_{1;k+n}} \\ B_{2;k+1} & B_{2;k+2} & \cdots & B_{2;k+n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{k;k+1} & \mathbf{B_{k;k+2}} & \cdots & B_{k;k+n} \\ 0 & B_{k+1;k+2} & \cdots & B_{k+1;k+n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{k+n-1;k+n} \end{pmatrix} \tag{2.1.4}$$

The columns of matrix (2.1.4) are labeled 1 through n, and the rows are labeled 0 through $k+n-1$.

Each term (2.1.3) corresponds to a unique matrix of form (2.1.4).

In the following section, we will present a few key lemmas to help us with the combinatorial reduction. For the proof of these lemmas, we refer the reader to [34]

2.2 Acceptable moves and equivalence classes

The next step in to introduce the relationship of equivalence in the set $\mathcal{M}_{k,n}$. In order to do that, authors of [34] introduce the notion of an *acceptable* *move* on the set $\mathcal{M}_{k,n}$.

Definition 2.1 (Acceptable moves)**.** *If $\mu(k+j+1) < \mu(k+j)$, we take the following steps at the same time*

- *exchange the highlights in columns j and $j+1$*
- *exchange the highlights in rows $k+j$ and $k+j+1$*
- *exchange $t_{\sigma^{-1}(j)}$ and $t_{\sigma^{-1}(j+1)}$*

The lemma below highlights the importance of acceptable moves. It states that when one performs an acceptable move relating (μ, σ) and (μ', σ'), the values of the corresponding integrals $I(\mu, \sigma)$ and $I(\mu', \sigma')$ are the same.

Lemma 2.2. *Let (μ, σ) be transformed into (μ', σ') by an acceptable move. Then, for the corresponding integrals* (2.1.3)*, we have $I(\mu, \sigma) = I(\mu', \sigma')$*

Consider the subset $\{\mu_s\} \subset \mathcal{M}_{k,n}$ of *special upper echelon* matrices in which each highlighted element of a higher row is to the left of each highlighted element of a lower row. An example of a special upper echelon matrix (with $k = 1, n = 4$) is

$$\begin{pmatrix} \mathbf{B_{1;2}} & \mathbf{B_{1;3}} & B_{1;4} & B_{1;5} \\ 0 & B_{2;3} & B_{2;4} & B_{2;5} \\ 0 & 0 & \mathbf{B_{3;4}} & B_{3;5} \\ 0 & 0 & 0 & \mathbf{B_{4;5}} \end{pmatrix}.$$

Then it can be shown (see [34]):

Lemma 2.3. *For each element of $\mathcal{M}_{k,n}$ there is a finite number of acceptable moves which brings the matrix to an upper echelon matrix.*

Lemma 2.4. *Let $C_{k,n}$ be the number of $(k+n-1) \times n$ special upper echelon matrices of the type discussed above. Then $C_{k,n} \leq 2^{k+2n-2}$.*

Let μ_s be a special upper echelon matrix. We say μ is in the *equivalence class* of μ_s: $\mu \sim \mu_s$ if μ can be transformed to μ_s in finitely many acceptable moves.

One treats matrices in the same equivalence class via the following theorem:

Theorem 2.1. *There exists a subset D of $[0,t]^n$ such that*

$$\sum_{\mu \sim \mu_s} \int_0^t \dots \int_0^{t_{n-1}} J^k(\underline{t}_n; \mu) dt_1 \dots dt_n = \int_{D_{\mu_s,t}} J^k(\underline{t}_n; \mu_s) dt_1 \dots dt_n. \tag{2.2.1}$$

With the above theorem, one is able to reduce the sum of $\mathcal{O}(n!)$ terms into a sum of $\mathcal{O}(C^n)$ terms:

$$\gamma^{(k)}(t) = \sum_{\mu_s \in \mathcal{M}_{k,n}} \int_{D_{\mu_s,t}} d\underline{t}_n J^k(\underline{t}_n; \mu_s). \tag{2.2.2}$$

Chapter 3

Derivation of a NLS with a general power-type nonlinearity in 1D and 2D

In this chapter we derive (from a quantum many body system) a NLS with a general power-type nonlinearity. More precisely, we consider a quantum many body system which models a finite linear combination of n-body interactions and we obtain that the k-particle marginal density of the corresponding BBGKY hierarchy converges when the particle number goes to infinity.

We showed a priori energy bound which allows us to extract converging subsequences from the solutions of the BBGKY. Moreover, we proved that the limit of the BBGKY solutions actually solves a corresponding GP hierarchy of equations. The convergence is established by adapting the arguments originated or developed in [14, 15, 16, 17, 32, 7]. We also proved the uniqueness of solution to the GP hierarchy based on a priori space time estimates. The main tools in the uniqueness proof is a generalized board game (see [34]) argument which is used to handle the factorial growth in the number of terms from Duhamel expansion. In such a way, we proved that the dynamics of the system is determined by a NLS with a linear combination of power-type nonlinearities.

3.1 BBGKY and GP

In this section we introduce BBGKY and GP hierarchies that are relevant for our derivation of the NLS with a general power-type nonlinearity.

3.1.1 BBGKY hierarchy

We consider a quantum mechanical system of N bosonic particles in $\mathbb{R}^d$, with $d \in \{1,2\}$. Let p and p_0 be positive integers, and p_0 is fixed with $1 \le p \le p_0$. The time evolution of the N-particle wave function $\psi_N \in L^2_s(\mathbb{R}^{dN})$ is governed by the Schrödinger equation:

$$i\partial_t \psi_N(t) = H_N \psi_N(t), \tag{3.1.1}$$

with the Hamiltonian (the index $i_1, \cdots, i_{p+1} \in \{1, 2, \cdots, N\}$)

$$H_N := \sum_{i=1}^{N}(-\Delta_{x_i}) + \sum_{p=1}^{p_0} \frac{1}{N^p} \sum_{i_1 < \cdots < i_{p+1}} N^{pd\beta} V^{(p)}\big(N^\beta(x_{i_1} - x_{i_2}), \cdots, N^\beta(x_{i_1} - x_{i_{p+1}})\big) \tag{3.1.2}$$

on Hilbert space $L^2_s(\mathbb{R}^{dN})$. $L^2_s(\mathbb{R}^{dN})$ is the subspace of $L^2(\mathbb{R}^{dN})$ consisting of all functions satisfying

$$\psi_N(x_{\sigma(1)}, x_{\sigma(2)} ..., x_{\sigma(N)}) = \psi_N(x_1, x_2 ..., x_N),$$

for any permutation $\sigma \in S_N$ and $0 < \beta < \frac{1}{2dp+2}$. Also we assume that for all $1 \le p \le p_0$ the $(p+1)$-body interaction potential $V^{(p)} \in W^{p,\infty}(\mathbb{R}^{pd})$ is a non-negative function with sufficient regularity and is translation-invariant so that it can be written in the above form. For instance, when $p = 2$, we have that

$$V^{(2)}(x_1 - x_2, x_2 - x_3, x_1 - x_3) = V^{(2)}(x_1 - x_2, -(x_1 - x_2) + (x_1 - x_3), x_1 - x_3)$$
$$\equiv V^{(2)}(x_1 - x_2, x_1 - x_3). \tag{3.1.3}$$

The first part of the Hamiltonian represents the kinetic energy, while the second is the sum of interaction potentials involving $p+1$ particles.

Note that (3.1.1) is linear, which together with the fact that H_N is a self-adjoint operator implies that global in time solutions can be written by means of the unitary group generated by H_N as

$$\psi_N(t) = e^{-iH_N t}\psi_N(0), \qquad \forall t \in \mathbb{R} \tag{3.1.4}$$

Let $V_N^{(p)}(x_1, x_2, \cdots, x_p) := N^{pd\beta} V^{(p)}(N^\beta x_1, N^\beta x_2, \cdots, N^\beta x_p)$ be the rescaled potential. Since $\psi_N(t)$ satisfies (3.1.1). We can verify that the marginal densities $\gamma_N^{(k)}(t)$ satisfy the following *BBGKY hierarchy*

$$\begin{aligned}
i\partial_t \gamma_N^{(k)}(t) = & \\
& \sum_{i=1}^{k} [-\Delta_{x_i}, \gamma_N^{(k)}(t)] \\
& + \sum_{p=1}^{p_0} \Bigg\{ \frac{1}{N^p} \sum_{1 \le i_1 < \cdots < i_{p+1} \le k} [V_N^{(p)}(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{p+1}}), \gamma_N^{(k)}(t)] \\
& \quad + \frac{N-k}{N^p} \sum_{1 \le i_1 < \cdots < i_p \le k} \mathrm{Tr}_{k+1}[V_N^{(p)}(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_p}, x_{i_1} - x_{k+1}), \gamma_N^{(k+1)}(t)] \\
& \quad + \frac{(N-k)(N-k-1)}{N^p} \sum_{1 \le i_1 < \cdots < i_{p-1} \le k} \mathrm{Tr}_{k+1}\mathrm{Tr}_{k+2} \\
& \qquad [V_N^{(p)}(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{p-1}}, x_{i_1} - x_{k+1}, x_{i_2} - x_{k+2}), \gamma_N^{(k+2)}(t)]
\end{aligned} \tag{3.1.5}$$

$$+\cdots$$
$$+\frac{(N-k)(N-k-1)\cdots(N-k-p+1)}{N^p}\sum_{1\le i_1\le k}\mathrm{Tr}_{k+1}\mathrm{Tr}_{k+2}\cdots\mathrm{Tr}_{k+p}$$
$$\left.[V_N^{(p)}(x_{i_1}-x_{k+1},x_{i_1}-x_{k+2},\cdots,x_{i_1}-x_{k+p}),\gamma_N^{(k+p)}(t)]\right\}.$$

Here we use the convention that $\gamma_N^{(k)}(t)=0$, whenever $k>N$. The symbol Tr_{k+j} denotes the partial trace over the m-th particle, i.e, the kernel of the k-particle operator $\mathrm{Tr}_{k+1}[V_N^{(p)}(x_{i_1}-x_{i_2},\cdots,x_{i_1}-x_{i_p},x_{i_1}-x_{k+1}),\gamma_N^{(k+1)}(t)]$ is given by

$$\begin{aligned}&\big(\mathrm{Tr}_{k+1}[V_N^{(p)}(x_{i_1}-x_{i_2},\cdots,x_{i_1}-x_{i_p},x_{i_1}-x_{k+1}),\gamma_N^{(k+1)}(t)]\big)(\mathbf{x}_k;\mathbf{x}'_k)\\&=\int V_N^{(p)}(x_{i_1}-x_{i_2},\cdots,x_{i_1}-x_{i_p},x_{i_1}-x_{k+1})\gamma^{(k+1)}(\mathbf{x}_k,x_{k+1};\mathbf{x}'_k,x_{k+1})dx_{k+1}\\&\quad-\int V_N^{(p)}(x'_{i_1}-x'_{i_2},\cdots,x'_{i_1}-x'_{i_p},x'_{i_1}-x_{k+1})\gamma^{(k+1)}(\mathbf{x}_k,x_{k+1};\mathbf{x}'_k,x_{k+1})dx_{k+1}.\end{aligned}\tag{3.1.6}$$

Let us present a heuristic argument on what one expects when taking $N\to\infty$. We note that all the terms in (3.1.5), except the first term on the RHS and the last term in the bracket, are expected to vanish for fixed k and sufficiently small β, because $\frac{1}{N^p}\to 0$, $\frac{\prod_{i=0}^{j}(N-k-i)}{N^p}\to 0$, $\forall 0\le j\le p-2$. The last interaction term on the RHS is expected to survive since $\frac{\prod_{i=0}^{p-1}(N-k-i)}{N^p}\to 1$. Indeed, one can make this heuristic precise and prove existence of a weak sequential limit of (3.1.5) under the same topology that was originally used in [15], and subsequently in [32, 7]. Details are presented in §3.3. In such a way one shows that the corresponding infinite (GP) hierarchy is a weak sequential limit of (3.1.5).

3.1.2 GP hierarchy

Following the convention in Chen-Pavlović [6], we formally write down the limit of (3.1.5) as $N \to \infty$, as follows:

$$i\partial_t \gamma^{(k)}(t) = \sum_{j=1}^{k} \big(-\Delta_{x_j} + \Delta_{x'_j}\big)\gamma^{(k)}(t) + \sum_{p=1}^{p_0} b_p \sum_{j=1}^{k} B_{j;k+1,\ldots,k+p}\gamma^{(k+p)}(t). \quad (3.1.7)$$

for any $k \geq 1$. We call (3.1.7) *cubic Gross-Pitaevskii (GP) hierarchy* if $p = 1$; *quintic GP hierarchy* if $p = 2$ and *septic GP hierarchy* if $p = 3$, and so on. Here b_p is the L^1 norm of the non-negative potential: $b_p = \int_{\mathbb{R}^{pd}} V^{(p)}(x_1, \cdots, x_p)dx_1 \cdots dx_p$.

The *contraction operator* is given via

$$B_{j;k+1,\cdots,k+p} := B^{+}_{j;k+1,\cdots,k+p} - B^{-}_{j;k+1,\cdots,k+p}. \quad (3.1.8)$$

where

$$\begin{aligned}
&\Big(B^{+}_{j;k+1,\cdots,k+p}\gamma^{(k+p)}\Big)(t, \mathbf{x}_k, \mathbf{x}'_k) \\
:= &\int \delta(x_j - x_{k+1})\delta(x_j - x'_{k+1}) \cdots \delta(x_j - x_{k+p})\delta(x_j - x'_{k+p}) \\
&\quad \times \gamma^{(k+p)}(t, x_1, \cdots, x_{k+p}; x'_1, \cdots, x'_{k+p})dx_{k+1}dx'_{k+1} \cdots dx_{k+p}dx'_{k+p}.
\end{aligned} \quad (3.1.9)$$

and

$$\begin{aligned}
&\Big(B^{-}_{j;k+1,\cdots,k+p}\gamma^{(k+p)}\Big)(t, \mathbf{x}_k, \mathbf{x}'_k) \\
:= &\int \delta(x'_j - x_{k+1})\delta(x'_j - x'_{k+1}) \cdots \delta(x'_j - x_{k+p})\delta(x'_j - x'_{k+p}) \\
&\quad \times \gamma^{(k+p)}(t, x_1, \cdots, x_{k+p}; x'_1, \cdots, x'_{k+p})dx_{k+1}dx'_{k+1} \cdots dx_{k+p}dx'_{k+p}.
\end{aligned} \quad (3.1.10)$$

We can check that

$$\gamma^{(k)} = |\phi_t\rangle \langle\phi_t|^{\otimes k} = \prod_{j=1}^{k} \phi_t(x_j)\overline{\phi}_t(x'_j) \tag{3.1.11}$$

is a solution to (3.1.7) if ϕ_t is a solution to the nonlinear Schrödinger equation

$$i\partial_t\phi_t = -\Delta\phi_t + \sum_{p=1}^{p_0} b_p|\phi_t|^{2p}\phi_t. \tag{3.1.12}$$

We establish the uniqueness of solutions to the GP hierarchy, and build the following convergence under appropriate topology:

$$\gamma_N^{(k)} \to \gamma^{(k)}, \qquad as \quad N \to \infty, \quad \forall k \geq 1. \tag{3.1.13}$$

3.2 Statement of the main result

Our main result for the chapter is the theorem below:

Theorem 3.1. *Let $p_0 \geq 1$ be a fixed integer. Suppose that for all $1 \leq p \leq p_0$ the potential $V^{(p)} \in W^{p,\infty}(\mathbb{R}^{dp})$ and $V^{(p)} \geq 0$ is symmetric and translation-invariant. Let $d \in \{1, 2\}$ and $0 < \beta < \frac{1}{2dp_0+2}$. $\{\psi_N\}_{N\geq1}$ is a family of functions that satisfy*

$$\sup_N \frac{1}{N}\langle\psi_N, H_N\psi_N\rangle < \infty \tag{3.2.1}$$

and assume $\{\psi_N\}_{N\geq1}$ exhibits asymptotic factorization: $\exists\phi \in L^2(\mathbb{R}^d)$ such that $\mathrm{Tr}\big|\gamma_N^{(1)} - |\phi\rangle\langle\phi|\,\big| \to 0$ as $N \to \infty$, where $\gamma_N^{(1)}$ is the 1-particle marginal density associated with ψ_N.

Then we have

$$\mathrm{Tr}\Big|\gamma_N^{(k)} - |\phi\rangle\langle\phi|^{\otimes k}\,\Big| \to 0 \quad as \quad N \to \infty. \tag{3.2.2}$$

Here, $\gamma_N^{(k)}$ is the k-particle marginal density associated to $\psi_N(t) = e^{-iH_N t}\psi_N(0)$, and $\phi(t)$ solves the nonlinear Schrödinger equation: $i\partial_t\phi(t) = -\Delta\phi(t) + \sum_{p=1}^{p_0} b_p|\phi(t)|^{2p}\phi(t)$ with initial condition $\phi(0) = \phi_0$ and potential constant $b_p = \int_{\mathbb{R}^{pd}} V^{(p)}(x)dx < \infty$.

The strategy we follow is to identify the limit of $\Gamma_N = \{\gamma_N^{(k)}\}_{k=1}^N$ as the unique solution to (3.1.7); or in other words, every limit (under suitable topology) of Γ_N solves (3.1.7) uniquely, since (3.1.11) is a solution, then (3.2.2) follows by compactness.

The idea to prove uniqueness of the infinite hierarchy in [34] consists of the following three steps. First, we express each solution $\gamma^{(k)}$ in terms of the future iterates $\gamma^{(k+p_0)}, ..., \gamma^{(k+np_0)}$ using Duhamel formula (we choose all p to be p_0 for a upper bound of the number of terms). Since for each p_0, the operator $B_{k+p_0}^k = \sum_{j=1}^k B_{j;k+1,\cdots,k+p_0 n}$ is a sum of k operators, the iterated Duhamel formula involves up to $k(k+p_0)\cdots\big(k+p_0(n-1)\big) \sim \mathcal{O}(n!)$ terms (see J^k in (3.6.2)). Then in the second step, we use a combinatorial argument to group these iterated terms into equivalence classes that we can bound. Finally, we treat each equivalence class with the Strichartz type estimate (3.5.4).

Compared to the previous work, the main novelties are:

- Due to the presence of different n-body interaction potentials, our proof of a priori energy bound (Proposition 3.1) involves more cases than the previous single potential model;

- In the combinatorial argument, the matrix associated with iterated Duhamel terms reflects a combination of different interactions, which results in a dynamical structure in the matrix rather than a fixed form;

- Our main theorem is for general many-body interactions without specifying the number of particles that are involved, so it gives complete answer to all questions of similar type.

3.3 Convergence

We prove the main theorem in this section. In §3.3.1, a useful priori energy bound is established. Based on which, we summarized the main steps of the proof of the compactness of the sequence of k-particle marginals and the convergence to the infinite hierarchy in §3.3.2. We put the proof of a approximation lemma in §3.3.3.

3.3.1 A priori energy bound

From the energy estimates, following [32, 7, 15, 13, 18], we obtained a priori bound below.

Proposition 3.1. *Suppose $0 < \beta < \frac{1}{2dp_0+2}$, then there exists a constant C (depends on $p_0, V^{(p)}, d$), such that for every k, there exists $N_0(k)$ such that*

$$\langle \psi, (H_N + N)^k \psi \rangle \geq C^k N^k \langle \psi, (1 - \Delta_{x_1}) \cdots (1 - \Delta_{x_k}) \psi \rangle \tag{3.3.1}$$

for all $N \geq N_0(k)$, $\psi \in L^2_s(\mathbb{R}^{dN})$. The Hamiltonian H_N is defined as in (3.1.2).

Proof. We adapt the proof in [32, 7] to the current case. It is a two-step induction over $k \geq 0$. For $k = 0$ the statement is trivial and for $k = 1$ the statement follows from $V_N^{(p)} \geq 0$. In order to illustrate the techniques here, we check one more case before proving the induction step. Write $S_i = (1 - \Delta_{x_i})^{\frac{1}{2}}$ and the interactions in two groups h_1 and h_2, such that $H_N + N = h_1 + h_2$:

$$h_1 = \sum_{j=n+1}^{N} S_j^2$$
$$h_2 = \sum_{j=1}^{n} S_j^2 + \sum_{p=1}^{p_0} \sum_{i_1 < i_2 < \cdots < i_{1+p}} N^{-p} V_N^{(p)}(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{1+p}}).$$

For $k = 2$, let $h_1 = \sum_{j=1}^{N} S_j^2$ and $h_2 = \sum_p \sum N^{-p} V_N^{(p)}(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{1+p}})$, then since $h_2^2 \geq 0$,

$$\begin{aligned}
&\langle \psi, (H_N + N)^2 \psi \rangle \\
&= \langle \psi, h_1^2 \psi \rangle + \langle \psi, h_1 h_2 \psi \rangle + \langle \psi, h_2 h_1 \psi \rangle + \langle \psi, h_2^2 \psi \rangle \\
&\geq \langle \psi, h_1^2 \psi \rangle + \langle \psi, h_1 h_2 \psi \rangle + \langle \psi, h_2 h_1 \psi \rangle \\
&= N(N-1) \langle \psi, S_1^2 S_2^2 \psi \rangle + N \langle \psi, S_1^4 \psi \rangle \qquad (\text{“leading terms”}) \\
&+ \sum_{p=1}^{p_0} N \sum N^{-p} (\langle \psi, S_1^2 V_N^{(p)}(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{1+p}}) \psi \rangle + c.c), (\text{“error terms”})
\end{aligned} \tag{3.3.2}$$

where c.c denotes “complex conjugate“. We keep the “leading terms” in RHS of (3.3.2) and look for a lower bound of the terms in the last line (“error terms”). As in [7], let $\dot{S}_j = (\dot{S}_{j,i})_{i=1}^d := i\nabla_{x_j}$, then $S_j^2 = 1 + \dot{S}_j^2 = 1 - \Delta_{x_j}$. For sufficiently large N, by the permutation symmetry of ψ:

$$N \sum N^{-p} (\langle \psi, S_1^2 V_N^{(p)}(x_{i_1} - x_{i_2}, \cdots, x_{i_1} - x_{i_{1+p}}) \psi \rangle + c.c)$$

$$
\begin{aligned}
&= N^{1-p}(N-1)\cdots(N-p-1)(\langle\psi, S_1^2 V_N^{(p)}(x_2-x_3,\cdots,x_2-x_{2+p})\psi\rangle + c.c)\\
&\quad + N^{1-p}(N-1)\cdots(N-p)(\langle\psi, S_1^2 V_N^{(p)}(x_1-x_2,\cdots,x_1-x_{1+p})\psi\rangle + c.c)\\
&\geq C^2N^2(\langle\psi, S_1^2 V_N^{(p)}(x_2-x_3,\cdots,x_2-x_{2+p})\psi\rangle + c.c)\\
&\quad + CN(\langle\psi, (1+\dot{S}_1^2) V_N^{(p)}(x_1-x_2,\cdots,x_1-x_{1+p})\psi\rangle + c.c)\\
&\geq CN(\langle\psi, \dot{S}_1^2 V_N^{(p)}(x_1-x_2,\cdots,x_1-x_{1+p})\psi\rangle + c.c) \qquad (3.3.3)\\
&\geq -CN\big|\langle\psi, \dot{S}_1\big(\nabla_{x_1} V_N^{(p)}(x_1-x_2,\cdots,x_1-x_{1+p})\big)\psi\rangle\big|\\
&\geq -CN\rho\big|\langle\psi, S_1^2\psi\rangle\big| - \frac{CN}{\rho}\big|\langle\psi, |\nabla_{x_1} V_N^{(p)}|^2\psi\big|\\
&\geq -CN\rho\big|\langle\psi, S_1^2\psi\rangle\big| - \frac{CN}{\rho}\|\nabla V_N^{(p)}\|^2_{L^\infty(\mathbb{R}^{dp})}\langle\psi, S_1^2S_2^2\psi\rangle \qquad (3.3.4)\\
&= -CN\rho\big|\langle\psi, S_1^2\psi\rangle\big| - \frac{CN^{1+(2pd+2)\beta}}{\rho}\|\nabla V^{(p)}\|^2_{L^\infty(\mathbb{R}^{dp})}\langle\psi, S_1^2S_2^2\psi\rangle.
\end{aligned}
$$

To obtain (3.3.3), we dropped positive terms using the positivity of $V_N^{(p)}$; $\rho > 0$ is arbitrary and we've applied Lemma 3.2 to obtain (3.3.4). Thus

$$
\begin{aligned}
&\langle\psi, (H_N+N)^2\psi\rangle\\
&\geq N(N-1)\langle\psi, S_1^2S_2^2\psi\rangle + N\langle\psi, S_1^4\psi\rangle\\
&\qquad - \sum_{p=1}^{p_0}\Big(CN\rho\big|\langle\psi, S_1^2\psi\rangle\big| + \frac{CN^{1+(2pd+2)\beta}}{\rho}\langle\psi, S_1^2S_2^2\psi\rangle\Big)\\
&\geq C^2N^2\langle\psi, S_1^2S_2^2\psi\rangle, \qquad \text{for big } N \text{ with } \beta < \frac{1}{2dp_0+2}.
\end{aligned}
$$

The basic idea in the proof is to derive a lower bound of the "error terms" which is dominated by the "leading terms". Now assume (3.3.1) is true for all $k \leq n$, then we prove it holds for $k = n+2$. For big enough N, by the induction assumption, we have (since $H_N + N$ is self-adjoint):

$$
\langle\psi, (H_N+N)^{n+2}\psi\rangle \geq C^nN^n\langle\psi, (H_N+N)S_1^2\cdots S_n^2(H_N+N)\psi\rangle. \qquad (3.3.5)
$$

Then it follows that

$$\begin{aligned}
&\langle\psi,(H_N+N)S_1^2\cdots S_n^2(H_N+N)\psi\rangle \\
&=\langle\psi,h_1S_1^2\cdots S_n^2h_1\psi\rangle+\langle\psi,h_1S_1^2\cdots S_n^2h_2\psi\rangle \\
&\quad+\langle\psi,h_2S_1^2\cdots S_n^2h_1\psi\rangle+\langle\psi,h_2S_1^2\cdots S_n^2h_2\psi\rangle.
\end{aligned} \tag{3.3.6}$$

Note that $h_2S_1^2\cdots S_n^2h_2\geq 0$. Combine (3.3.5) and (3.3.6) and use the permutation symmetry of ψ to get:

$$\begin{aligned}
&\langle\psi,(H_N+N)^{n+2}\psi\rangle \\
&\geq C^nN^n\big(\langle\psi,h_1S_1^2\cdots S_n^2h_1\psi\rangle+\langle\psi,h_1S_1^2\cdots S_n^2h_2\psi\rangle+\langle\psi,h_2S_1^2\cdots S_n^2h_1\psi\rangle\big) \\
&\geq C^nN^n(N-n)\Big\{(N-n-1)\langle\psi,S_1^2\cdots S_{n+2}^2\psi\rangle+n\langle\psi,S_1^4S_2^2\cdots S_{n+1}^2\psi\rangle \\
&\quad+\sum_{p=1}^{p_0}\frac{1}{N^p}\sum_{i_1<\cdots<i_{1+p}}\big(\langle\psi,S_1^2\cdots S_{n+1}^2V_N^{(p)}(x_{i_1}-x_{i_2},\cdots,x_{i_1}-x_{i_{1+p}})\psi\rangle+c.c\big)\Big\}.
\end{aligned} \tag{3.3.7}$$

The last term above is the error term we want to control. Again by permutation symmetry of ψ, we can further break down the interactions of the last term in (3.3.7) for big enough N:

$$\begin{aligned}
&\langle\psi,(H_N+N)^{n+2}\psi\rangle \\
&\geq C^{n+2}N^{n+2}\langle\psi,S_1^2\cdots S_{n+2}^2\psi\rangle+C^{n+1}N^{n+1}\big(\langle\psi,S_1^4\cdots S_{n+1}^2\psi\rangle \qquad (3.3.8)\\
&\quad+\sum_{p=1}^{p_0}C^nN^{n-p}(N-n)(N-n-1)\cdots(N-n-1-p) \qquad (3.3.9)\\
&\qquad\times\langle\psi,S_1^2\cdots S_{n+1}^2V_N^{(p)}(x_{n+2}-x_{n+3},\cdots,x_{n+2}-x_{n+2+p})\psi\rangle \\
&\quad+\sum_{p=1}^{p_0}\sum_{j=2}^{1+p}C^nN^{n-p}(N-n)(N-n-1)\cdots(N-n-p+j-2) \\
&\qquad\times(n+1)n\cdots(n+3-j)\langle\psi,S_1^2\cdots S_{n+1}^2V_N^{(p)}(x_1-x_2,\cdots,x_1-x_{j-1},
\end{aligned}$$

$$x_1 - x_{n+2}, \cdots, x_1 - x_{n+3+p-j})\psi\rangle \tag{3.3.10}$$

$$+ \sum_{p=1}^{p_0} C^n N^{n-p}(N-n)(n+1)n \cdots (n+1-p) \tag{3.3.11}$$

$$\times \langle \psi, S_1^2 \cdots S_{n+1}^2 V_N^{(p)}(x_1 - x_2, x_1 - x_3, \cdots, x_1 - x_{n+1}, \cdots, x_1 - x_{1+p})\psi\rangle.$$

We split terms as follows: (3.3.9)–(3.3.11): we put the "first" n particles in group h_2 and the "rest" in group h_1. Then the term (3.3.9) comes exclusively from group h_1 interactions; and term (3.3.11) is contributed purely by group h_2 interactions; (3.3.10) are mixture of inter-group and inner-group (h_2) interactions. We will handle each of these terms individually.

Our goal is to show that (3.3.9)–(3.3.11) are dominated by (3.3.8). Since p_0 is a finite number and N can be arbitrarily large, thus it suffices to show the goal for a single p with $1 \le p \le p_0$.

First of all, term (3.3.9) is non-negative and thus can be dropped for purpose of a lower bound. To see this, note $V_N^{(p)} \ge 0$ and commutes with all derivatives $S_1, S_2, \cdots, S_{n+1}$, we have

$$\begin{aligned}
&\langle \psi, S_1^2 \cdots S_{n+1}^2 V_N^{(p)}(x_{n+2} - x_{n+3}, \cdots, x_{n+2} - x_{n+2+p})\psi\rangle \\
&\quad = \int d\mathbf{x}_N V_N^{(p)}(x_{n+2} - x_{n+3}, \cdots, x_{n+2} - x_{n+2+p}) \left|(S_1 \cdots S_{n+1}\psi)(\mathbf{x}_N)\right|^2 \ge 0.
\end{aligned}$$

For (3.3.10), the sum over j consists of p terms (if $1 + p > n + 1$, (3.3.10) is a sum of n terms, and (3.3.11) vanishes). Consider the first term which corresponding to $j = 2$:

$$\langle \psi, S_1^2 \cdots S_{n+1}^2 V_N^{(p)}(x_1 - x_{n+2}, x_1 - x_{n+3}, \cdots, x_1 - x_{n+1+p})\psi\rangle$$

$$
\begin{aligned}
&\geq \langle \psi, S_{n+1}\cdots S_2 V_N^{(p)}(x_1 - x_{n+2}, x_1 - x_{n+3}, \cdots, x_1 - x_{n+1+p}) S_2 \cdots S_{n+1}\psi\rangle \\
&\quad - \left|\langle \psi, S_{n+1}\cdots S_2 \dot{S}_1 \big(\nabla_{x_1} V_N^{(p)}(x_1 - x_{n+2}, \cdots, x_1 - x_{n+1+p})\big) S_2 \cdots S_{n+1}\psi\rangle\right| \\
&\geq -\left|\langle \psi, S_{n+1}\cdots S_2 \dot{S}_1 \big(\nabla_{x_1} V_N^{(p)}(x_1 - x_{n+2}, \cdots, x_1 - x_{n+1+p})\big) S_2 \cdots S_{n+1}\psi\rangle\right| \qquad (3.3.12) \\
&\geq -\rho\left|\langle \psi, S_{n+1}^2 \cdots S_1^2 \psi\rangle\right| \\
&\quad - \frac{1}{\rho}\left|\langle \psi, S_{n+1}\cdots S_2 \left|\nabla_{x_1} V_N^{(p)}(x_1 - x_{n+2}, \cdots, x_1 - x_{n+1+p})\right|^2 S_2 \cdots S_{n+1}\psi\rangle\right| \qquad (3.3.13) \\
&\geq -\rho\left|\langle \psi, S_{n+1}^2 \cdots S_1^2 \psi\rangle\right| - \frac{1}{\rho}\left\|\nabla_{x_1} V_N^{(p)}\right\|_{L^\infty(\mathbb{R}^{dp})}^2 \langle \psi, S_1^2 \cdots S_{n+2}^2 \psi\rangle \qquad (3.3.14) \\
&= -\rho\left|\langle \psi, S_{n+1}^2 \cdots S_1^2 \psi\rangle\right| - \frac{CN^{(2pd+2)\beta}}{\rho}\langle \psi, S_1^2 \cdots S_{n+2}^2 \psi\rangle.
\end{aligned}
$$

which are dominated by the leading terms in (3.3.8) when $\beta < \frac{1}{2pd+2}$ (which is fine since p is at most p_0). The constant C depends on $\left\|\nabla_{x_1} V^{(p)}\right\|_{L^\infty(\mathbb{R}^{dp})}^2$. Here we use the positivity of $V_N^{(p)}$ to obtain (3.3.12). Note $\dot{S}_j^2 = S_j^2 - 1 < S_j^2$, $\rho > 0$ in (3.3.13) can be chosen arbitrarily, and in (3.3.14) we have applied (3.3.22) with $l = 2$.

For the term corresponding to $j = 3$ in (3.3.10):

$$
\begin{aligned}
&\langle \psi, S_1^2 \cdots S_{n+1}^2 V_N^{(p)}(x_1 - x_2, x_1 - x_{n+2}, \cdots, x_1 - x_{n+p})\psi\rangle \\
&\geq \langle \psi, S_{n+1}\cdots S_3 V_N^{(p)}(x_1 - x_2, x_1 - x_{n+2}, \cdots, x_1 - x_{n+p}) S_3 \cdots S_{n+1}\psi\rangle \qquad (3.3.15) \\
&+ \langle \psi, S_{n+1}\cdots S_3 (\dot{S}_1^2 + \dot{S}_2^2) V_N^{(p)}(x_1 - x_2, x_1 - x_{n+2}, \cdots, x_1 - x_{n+p}) S_3 \cdots S_{n+1}\psi\rangle \qquad (3.3.16) \\
&+ \langle \psi, S_{n+1}\cdots S_3 \dot{S}_2 \dot{S}_1 [\dot{S}_1 \dot{S}_2, V_N^{(p)}(x_1 - x_2, x_1 - x_{n+2}, \cdots, x_1 - x_{n+p})] S_3 \cdots S_{n+1}\psi\rangle. \qquad (3.3.17)
\end{aligned}
$$

We know (3.3.15) is positive and thus can be discarded for a lower bound.

(3.3.16) can be treated as in the case $j = 2$. Note that

$$[\dot{S}_1 \dot{S}_2, V_N^{(p)}] = [\dot{S}_1, V_N^{(p)}]\dot{S}_2 + \dot{S}_1[\dot{S}_2, V_N^{(p)}]$$

Hence

$$\begin{aligned}
&(3.3.17)\\
&\geq -\big|\langle \psi, S_{n+1} \cdots S_3 \dot{S}_2 \dot{S}_1\\
&\qquad [\dot{S}_1, V_N^{(p)}(x_1 - x_2, x_1 - x_{n+2}, \cdots, x_1 - x_{n+p})]\dot{S}_2 S_3 \cdots S_{n+1}\psi\rangle\big|\\
&\quad - \big|\langle \psi, S_{n+1} \cdots S_3 \dot{S}_2 \dot{S}_1^2\\
&\qquad [\dot{S}_2, V_N^{(p)}(x_1 - x_2, x_1 - x_{n+2}, \cdots, x_1 - x_{n+p})] S_3 \cdots S_{n+1}\psi\rangle\big|\\
&\geq -\rho_1\big|\langle \psi, S_{n+1}^2 \cdots S_2^2 S_1^2 \psi\big| - \frac{1}{\rho_1}\|\nabla V_N^{(p)}\|^2_{L^\infty(\mathbb{R}^{dp})}\langle \psi, S_1^2 \cdots S_{n+2}^2 \psi\rangle\\
&\quad - \rho_2\big|\langle \psi, S_{n+1}^2 \cdots S_2^2 S_1^4 \psi\big| - \frac{1}{\rho_2}\|V_N^{(p)}\|^2_{W^{1,\infty}(\mathbb{R}^{dp})}\langle \psi, S_1^2 \cdots S_{n+1}^2 \psi\rangle.
\end{aligned}$$

We shall prove the estimate for general terms in (3.3.10) by running a one-step induction in j. Note that the j-th term T_j in (3.3.10), with $2 \leq j \leq 1 + p$, has the coefficient of order $O(N^{n-j+3})$. Assume we have the desired bound for j from 2 through j_0, that is

$$\begin{aligned}
&T_2 \geq -(CN)^{n+1+(2pd+2)\beta}\langle \psi, S_1^2 \cdots S_{n+2}^2 \psi\rangle,\\
&T_3 \geq -(CN)^{n+(2pd+2)\beta}\langle \psi, S_1^2 \cdots S_{n+2}^2 \psi\rangle,\\
&\cdots\\
&T_{j_0} \geq -(CN)^{n-j_0+3+\delta_{j_0}(\beta)}\langle \psi, S_1^2 \cdots S_{n+2}^2 \psi\rangle.
\end{aligned}$$

Function $\delta_j(\beta)$ ($2 \leq j \leq j_0$) take values in interval $(0, 1)$, this small power on N is contributed by appropriate norm of $V_N^{(p)}$. By the cases we have already checked, we know that $j_0 \geq 3$. Rewrite the main part of T_{j_0+1} as the following

$$\begin{aligned}
&\langle \psi, S_1^2 \cdots S_{n+1}^2 V_N^{(p)}(x_1 - x_2, \cdots, x_1 - x_{j_0}, x_1 - x_{n+2}, \cdots, x_1 - x_{n+3+p-j_0-1})\psi\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.3.18) \\
&= \langle \psi, (1 + \dot{S}_1^2) \cdots (1 + \dot{S}_{j_0}^2) S_{j_0+1}^2 \cdots S_{n+1}^2 V_N^{(p)} \psi\rangle \\
&= \langle \psi, S_{n+1} \cdots S_{j_0+1} V_N^{(p)} S_{j_0+1} \cdots S_{n+1}\psi\rangle \\
&\quad + \sum_{1 \leq r \leq j_0} \langle \psi, \dot{S}_r^2 S_{j_0+1}^2 \cdots S_{n+1}^2 V_N^{(p)} \psi\rangle \\
&\quad + \sum_{1 \leq r_1 < r_2 \leq j_0} \langle \psi, \dot{S}_{r_1}^2 \dot{S}_{r_2}^2 S_{j_0+1}^2 \cdots S_{n+1}^2 V_N^{(p)} \psi\rangle \\
&\quad + \cdots \\
&\quad + \sum_{1 \leq r \leq j_0} \langle \psi, \dot{S}_1^2 \cdots \hat{\dot{S}}_r^2 \cdots \dot{S}_{j_0}^2 S_{j_0+1}^2 \cdots S_{n+1}^2 V_N^{(p)} \psi\rangle \\
&\quad + \langle \psi, \dot{S}_1^2 \dot{S}_2^2 \cdots \dot{S}_{j_0}^2 S_{j_0+1}^2 \cdots S_{n+1}^2 V_N^{(p)} \psi\rangle,
\end{aligned}$$

where a hat denotes a missing term. Thanks to the induction assumption we may conclude that the lower bounds of all the terms in the RHS of (3.3.18) are controlled by the leading terms in (3.3.8) except the last term. By the definition of $\dot{S}_j$, we can prove the following decomposition:

$$\begin{aligned}
[\dot{S}_1 \cdots \dot{S}_{j-1}, V_N^{(p)}] = [\dot{S}_1, V_N^{(p)}]\dot{S}_2 \cdots \dot{S}_{j-1} + \dot{S}_1[\dot{S}_2, V_N^{(p)}]\dot{S}_3 \cdots \dot{S}_{j-1} + \cdots \\
+ \cdots + \dot{S}_1 \cdots \dot{S}_{j-2}[\dot{S}_{j-1}, V_N^{(p)}].
\end{aligned}$$

Therefore

$$\begin{aligned}
&\langle \psi, \dot{S}_1^2 \dot{S}_2^2 \cdots \dot{S}_{j_0}^2 S_{j_0+1}^2 \cdots S_{n+1}^2 V_N^{(p)} \psi\rangle \\
&= \langle \psi, S_{n+1} \cdots S_{j_0+1} \dot{S}_{j_0} \cdots \dot{S}_1 [\dot{S}_1 \cdots \dot{S}_{j_0}, V_N^{(p)}] S_{j_0+1} \cdots S_{n+1}\psi\rangle \\
&= \langle \psi, S_{n+1} \cdots S_{j_0+1} \dot{S}_{j_0} \cdots \dot{S}_1 ([\dot{S}_1, V_N^{(p)}]\dot{S}_2 \cdots \dot{S}_{j_0}) S_{j_0+1} \cdots S_{n+1}\psi\rangle
\end{aligned}$$

$$+\langle\psi, S_{n+1}\cdots S_{j_0+1}\dot{S}_{j_0}\cdots\dot{S}_2\dot{S}_1(\dot{S}_1[\dot{S}_2, V_N^{(p)}]\dot{S}_3\cdots\dot{S}_{j_0})S_{j_0+1}\cdots S_{n+1}\psi\rangle$$
$$+\cdots$$
$$+\langle\psi, S_{n+1}\cdots S_{j_0+1}\dot{S}_{j_0}\cdots\dot{S}_1(\dot{S}_1\cdots\dot{S}_{j_0-2}[\dot{S}_{j_0-1}, V_N^{(p)}]\dot{S}_{j_0})S_{j_0+1}\cdots S_{n+1}\psi\rangle$$
$$+\langle\psi, S_{n+1}\cdots S_{j_0+1}\dot{S}_{j_0}\cdots\dot{S}_1(\dot{S}_1\cdots\dot{S}_{j_0-1}[\dot{S}_{j_0}, V_N^{(p)}])S_{j_0+1}\cdots S_{n+1}\psi\rangle.$$

Again, by induction assumption all terms in the RHS of the above are bounded as we need except the one in the last line. However, we can reduce it into previous case since (for $j \geq 4$):

$$\dot{S}_1\cdots\dot{S}_{j-2}[\dot{S}_{j-1}, V_N^{(p)}] = \dot{S}_1\cdots\dot{S}_{j-3}[\dot{S}_{j-1}, (\dot{S}_{j-2}V_N^{(p)})] + \dot{S}_1\cdots\dot{S}_{j-3}[\dot{S}_{j-1}, V_N^{(p)}]\dot{S}_{j-2}. \tag{3.3.19}$$

Then

$$\begin{aligned}
&\langle\psi, S_{n+1}\cdots S_{j_0+1}\dot{S}_{j_0}\cdots\dot{S}_1(\dot{S}_1\cdots\dot{S}_{j_0-1}[\dot{S}_{j_0}, V_N^{(p)}])S_{j_0+1}\cdots S_{n+1}\psi\rangle \\
&= \langle\psi, S_{n+1}\cdots S_{j_0+1}\dot{S}_{j_0}\cdots\dot{S}_1\big(\dot{S}_1\cdots\dot{S}_{j_0-2}(\nabla_{j_0}\nabla_{j_0-1}V_N^{(p)})\big)S_{j_0+1}\cdots S_{n+1}\psi\rangle \\
&\quad + \langle\psi, S_{n+1}\cdots S_{j_0+1}\dot{S}_{j_0}\cdots\dot{S}_1\big(\dot{S}_1\cdots\dot{S}_{j_0-2}(i\nabla_{j_0}V_N^{(p)})\dot{S}_{j_0-1}\big)S_{j_0+1}\cdots S_{n+1}\psi\rangle.
\end{aligned}$$

Both terms above appear in the previous induction, but with one order higher derivative on $V_N^{(p)}$. Since we have

$$\|V_N^{(p)}\|^2_{W^{j_0-1,\infty}(\mathbb{R}^{dp})} \sim N^{2(pd\beta+(j_0-1)\beta)}\|V^{(p)}\|^2_{W^{j_0-1,\infty}(\mathbb{R}^{dp})},$$

we may set $\delta_{j_0}(\beta) = 2pd\beta + 2(j_0-1)\beta < 1$ (with $j_0 \geq 3$ since (3.3.19) requires $j \geq 4$). In general, the j-th term T_j in (3.3.10) has the following bound:

$$T_j \geq -N^{n-j+3}N^{2(pd\beta+(j-2)\beta)}\langle\psi, S_1^2\cdots S_{n+2}^2\psi\rangle, \forall V_N^{(p)} \in W^{j-2,\infty}, 3 \leq j \leq 1+p. \tag{3.3.20}$$

And $T_2 \geq -N^{n+1+(2dp+2)\beta}\langle\psi, S_1^2\cdots S_{n+2}^2\psi\rangle$. Admissible value for β will not send the total power of N to be greater than or equal to $n+2$. Thus for each $p \leq p_0$, β can take values in $(0, \frac{1}{2dp+2})$, which is actually determined by the base case $j = 2$.

Finally, the term (3.3.11) is actually a special case in (3.3.10) corresponding to $j = 2+p$, thus can be handled as above (the highest regularity of the potential is used here). This completes the proof. □

Lemma 3.2. *For $d \geq 1$, $m \geq 1$ and $\psi \in L_s^2(\mathbb{R}^{md})$, we have*

$$\langle\psi, V(x_1,\cdots,x_m)\psi\rangle \leq \|V\|_{L^r_{x_1,\cdots,x_m}}\langle\psi,(1-\Delta_{x_1})\cdots(1-\Delta_{x_m})\psi\rangle \tag{3.3.21}$$

for any $r > 2$ if $d \leq \frac{2}{m}$, and for $r \geq md$ if $d > \frac{2}{m}$. Moreover for any $1 \leq l \leq m$, we have

$$\langle\psi, V(x_1,\cdots,x_m)\psi\rangle \leq \|V\|_{L^\infty_{x_1,\cdots,x_m}}\langle\psi,\prod_{j=1}^{l}(1-\Delta_{x_j})\psi\rangle_{L^2_{x_1,\cdots,x_m}}. \tag{3.3.22}$$

Proof. By Hölder inequality with $\frac{1}{q}+\frac{1}{r}+\frac{1}{2} = 1$ and Sobolev embedding we have

$$\begin{aligned}
&\langle\psi, V(x_1,\cdots,x_m)\psi\rangle\\
\leq\ &\|V\|_{L^r_{x_1,\cdots,x_m}}\|\psi\|_{L^2_{x_1,\cdots,x_m}}\|\psi\|_{L^q_{x_1,\cdots,x_m}}\\
\leq\ &\|V\|_{L^r_{x_1,\cdots,x_m}}\|\psi\|_{L^2_{x_1,\cdots,x_m}}\|\psi\|_{H^1_{x_1,\cdots,x_m}}\\
\leq\ &\|V\|_{L^r_{x_1,\cdots,x_m}}\|\psi\|^2_{H^1_{x_1,\cdots,x_m}}\\
=\ &\|V\|_{L^r_{x_1,\cdots,x_m}}\|(1+|\xi_1|^2+|\xi_2|^2+\cdots+|\xi_m|^2)^{\frac{1}{2}}\hat{\psi}\|^2_{L^2}\\
\leq\ &\|V\|_{L^r_{x_1,\cdots,x_m}}\|(1+|\xi_1|^2)^{\frac{1}{2}}(1+|\xi_2|^2)^{\frac{1}{2}}\cdots(1+|\xi_m|^2)^{\frac{1}{2}}\hat{\psi}\|^2_{L^2}
\end{aligned}$$

$$= \|V\|_{L^r_{x_1,\cdots,x_m}} \langle \psi, (1-\Delta_{x_1})\cdots(1-\Delta_{x_m})\psi\rangle.$$

The Sobolev embedding requires that q is finite and satisfying $2 \le q \le \frac{2md}{md-2}$, which is equivalent to $2 \le q \le \frac{2md}{md-2}$ when $d > \frac{2}{m}$ and $2 \le q < \infty$ when $d \le \frac{2}{m}$. From the Hölder conjugate relations $\frac{1}{r} = \frac{1}{2} - \frac{1}{q}$, we know the constraints on r must be $r > 2$ if $d \le \frac{2}{m}$ and $r \ge md$ if $d > \frac{2}{m}$.

To prove (3.3.22), choose $q = 2, r = \infty$ in the above proof, then replace L^2 norm by H^1 norm in the first l variables to obtain:

$$\begin{aligned}
&\langle \psi, V(x_1,\cdots,x_m)\psi\rangle \\
&\le \|V\|_{L^\infty_{x_1,\cdots,x_m}} \|\psi\|^2_{L^2_{x_1,\cdots,x_m}} \\
&\le \|V\|_{L^\infty_{x_1,\cdots,x_m}} \|\psi\|^2_{H^1_{x_1,\cdots,x_l} L^2_{x_{l+1},\cdots,x_m}} \\
&= \|V\|_{L^\infty_{x_1,\cdots,x_m}} \|(1+|\xi_1|^2+|\xi_2|^2+\cdots+|\xi_l|^2)^{\frac{1}{2}}\hat{\psi}\|^2_{L^2_{\xi_1,\cdots,\xi_l} L^2_{x_{l+1},\cdots,x_m}} \\
&\le \|V\|_{L^\infty_{x_1,\cdots,x_m}} \|(1+|\xi_1|^2)^{\frac{1}{2}}(1+|\xi_2|^2)^{\frac{1}{2}}\cdots(1+|\xi_l|^2)^{\frac{1}{2}}\hat{\psi}\|^2_{L^2_{\xi_1,\cdots,\xi_l} L^2_{x_{l+1},\cdots,x_m}} \\
&= \big\|V\big\|_{L^\infty_{x_1,\cdots,x_m}} \langle \psi, (1-\Delta_{x_1})\cdots(1-\Delta_{x_l})\psi\rangle_{L^2_{x_1,\cdots,x_m}}.
\end{aligned}$$

Here the Fourier transform and its inverse transform of ψ are taken only on the first l variables with $1 \le l \le m$. □

After regularization of the initial data, we have

Corollary 3.3 (A priori bound)**.** *Let χ be a bump function with support on $[0,1]$ and $\kappa > 0$. Define*

$$\tilde{\psi}_N := \frac{\chi(\frac{\kappa}{N}H_N)\psi_N}{\big\|\chi(\frac{\kappa}{N}H_N)\psi_N\big\|}. \tag{3.3.23}$$

Let $\tilde{\psi}_N(t) = e^{-itH_N}\tilde{\psi}_N(0)$ and $\tilde{\gamma}_N^{(k)}$ be the corresponding k-marginal density. Then there exists a constant $\tilde{C} > 0$ depending on $\kappa, p_0, V^{(p)}$ for all $1 \leq p \leq p_0$ but independent of k, t, and there exists an integer $N_0(k)$ for every $k \geq 1$, such that for all $N > N_0(k)$, we have

$$\operatorname{Tr}(1-\Delta_{x_1})\cdots(1-\Delta_{x_k})\tilde{\gamma}_N^{(k)} \leq \tilde{C}^k. \tag{3.3.24}$$

Proof. The proof is simple when we have Proposition 3.1, since we have

$$\begin{aligned}
\operatorname{Tr}(1-\Delta_{x_1})\cdots(1-\Delta_{x_k})\tilde{\gamma}_N^{(k)}(t) &= \langle \tilde{\psi}_N^{(k)}(t), S_1^2 \cdots S_k^2 \tilde{\psi}_N^{(k)}(t)\rangle \\
&\leq \frac{1}{C^k N^k}\langle \tilde{\psi}_N^{(k)}(t), (H_N+N)^k \tilde{\psi}_N^{(k)}(t)\rangle \\
&\leq \frac{1}{C^k N^k}\langle \tilde{\psi}_N^{(k)}(t), 2^k (H_N^k+N^k) \tilde{\psi}_N^{(k)}(t)\rangle \\
&= \frac{2^k}{C^k N^k}\langle \tilde{\psi}_N^{(k)}(t), H_N^k \tilde{\psi}_N^{(k)}(t)\rangle + \frac{2^k}{C^k}\|\tilde{\psi}_N^{(k)}(t)\|^2 \\
&= \frac{2^k}{C^k N^k}\langle \tilde{\psi}_N^{(k)}(t), H_N^k \tilde{\psi}_N^{(k)}(t)\rangle + \frac{2^k}{C^k} \\
&\leq \tilde{C}^k.
\end{aligned} \tag{3.3.25}$$

In the first inequality we use Proposition 3.1, and in the last inequality we use the fact that $\langle \tilde{\psi}_N^{(k)}, H_N^k \tilde{\psi}_N^{(k)}\rangle \leq C^k N^k$ with the constant C depending on κ (see Proposition 5.1 in [15]). □

3.3.2 Compactness and convergence

The compactness of the k-particle marginal density sequence and the convergence to the infinite hierarchy are similar to the arguments in [15, 32, 7], we adapt the methods therein and outline the main steps in this section for completeness.

We introduce the following Banach spaces of density matrices. Denote by $\mathcal{K}_k = \mathcal{K}(L^2(\mathbb{R}^{dk}))$ the space of compact operators on $L^2(\mathbb{R}^{dk})$, equipped with the operator norm topology. And let $\mathcal{L}_k^1 = \mathcal{L}^1(L^2(\mathbb{R}^{dk}))$ denote the space of trace operators on $L^2(\mathbb{R}^{dk})$ equipped with the trace class norm. Then we know (see Theorem VI.26 in the book of Reed and Simon [44] for details)

$$\mathcal{L}_k^1 = \mathcal{K}_k^*. \tag{3.3.26}$$

The closed unit ball in $\mathcal{L}_k^1$ is weak* compact by Banach-Alaoglu theorem, and thus is metrizable in the weak* topology. Since $\mathcal{K}_k$ is separable, there exists a sequence $\{J_i^{(k)}\}_{i\geq 1} \in \mathcal{K}_k$, with $\|J_i^{(k)}\| \leq 1$, dense in the unit ball of $\mathcal{K}_k$. Then

$$\eta_k(\gamma^{(k)}, \tilde{\gamma}^{(k)}) := \sum_{i=1}^{\infty} 2^{-i} \big| \mathrm{Tr} J_i^{(k)}(\gamma^{(k)} - \tilde{\gamma}^{(k)}) \big| \tag{3.3.27}$$

is a metric on $\mathcal{L}_k^1$, and the induced topology by η_k is equivalent to the weak* topology on any weak* compact subset of $\mathcal{L}_k^1$ (Theorem 3.16 in Rudin's book [47]). Therefore a uniformly bounded sequence $\gamma_N^{(k)} \in \mathcal{L}_k^1$ converges to $\gamma^{(k)} \in \mathcal{L}_k^1$ with respect to the weak* topology if and only if $\eta_k(\gamma_N^{(k)}, \gamma^{(k)}) \to 0$ as $N \to \infty$. Now fix $T > 0$, let $C([0,T], \mathcal{L}_k^1)$ be the space of $\mathcal{L}_k^1$-valued functions of $t \in [0,T]$ which are continuous with respect to the metric η_k. We define the following metric $\hat{\eta}_k$ on $C([0,T], \mathcal{L}_k^1)$ for $k \in \mathbb{N}$:

$$\hat{\eta}_k(\gamma^{(k)}(\cdot), \tilde{\gamma}^{(k)}(\cdot)) := \sup_{t\in[0,T]} \eta_k(\gamma^{(k)}(t), \tilde{\gamma}^{(k)}(t)). \tag{3.3.28}$$

This induces the product topology τ_{prod} on $\bigoplus_{k\in\mathbb{N}} C([0,T], \mathcal{L}_k^1)$.

Proposition 3.4. *Let $\tilde{\psi}_N$ be defined as in* (3.3.23). *Then the sequence of marginal densities $\tilde{\Gamma}_N = \{\tilde{\gamma}_N^{(k)}\}_{k=1}^N \in \bigoplus_{k\in\mathbb{N}} C([0,T],\mathcal{L}_k^1)$ is compact with respect to the product topology τ_{prod} generated by the metric $\hat{\eta}_k$. If $\Gamma_{\infty,t} = \{\gamma^{(k)}(t)\}_{k\geq 1}$ is an arbitrary subsequential limit point, then its component $\gamma^{(k)}$ is non-negative and symmetric under permutations, and*

$$\mathrm{Tr}\gamma^{(k)} \leq 1$$

for every $k \geq 1$.

Sketch of the proof. Inspired by the proofs in [32, 7, 15], we notice that by Cantor's diagonal argument, it suffices to prove the compactness of $\tilde{\gamma}_N^{(k)}$ for some fixed k. Thanks to Arzelà-Ascoli theorem, this can be done by showing the equicontinuity of $\tilde{\gamma}_N^{(k)}$ with respect to the metric $\hat{\eta}_k$. Then it is enough to show that for every observable $J^{(k)}$ from a dense subset of $\mathcal{K}_k$ and for every $\epsilon > 0$, there exists $\delta = \delta(J^{(k)}, \epsilon)$ such that

$$\sup_{N\geq 1} \left|\mathrm{Tr}J^{(k)}(\tilde{\gamma}_N^{(k)}(t) - \tilde{\gamma}_N^{(k)})(s)\right| < \epsilon \tag{3.3.29}$$

for all $t, s \in [0,T]$ with $|t-s| \leq \delta$.

In order to prove (3.3.29), use (3.1.5) to rewrite $\tilde{\gamma}_N^{(k)}(t) - \tilde{\gamma}_N^{(k)}(s)$ in integral form and bound $\left|TrJ^{(k)}(\gamma_N^{(k)}(t) - \tilde{\gamma}_N^{(k)}(t))\right|$, which consists of $p+2$ terms, by the following:

$$\sup_{N\geq 1} \left|\mathrm{Tr}J^{(k)}(\tilde{\gamma}_N^{(k)}(t) - \tilde{\gamma}_N^{(k)}(s))\right| \leq C \left|\!\left|\!\left| J^{(k)} \right|\!\right|\!\right| |t-s|. \tag{3.3.30}$$

For this purpose, [7, 32] introduced an operator norm:

$$\left\|\left| J^{(k)} \right|\right\| := \sup_{\mathbf{p}'_k} \int d\mathbf{p}_k \prod_{j=1}^{k} \langle p_j \rangle \langle p'_j \rangle \left(|\hat{J}^{(k)}(\mathbf{p}_k; \mathbf{p}'_k)| + |\hat{J}^{(k)}(\mathbf{p}'_k; \mathbf{p}_k)| \right), \quad (3.3.31)$$

where $\hat{J}^{(k)}(\mathbf{p}_k; \mathbf{p}'_k)$ denotes the kernel of the compact operator $J^{(k)}$ in momentum space. Then use the fact that the set of all $J^{(k)} \in \mathcal{K}_k$ with finite norm is dense in $\mathcal{K}_k$ to reach the conclusion. □

From proposition 3.4, we know that the sequence $\tilde{\Gamma}_N = \{\tilde{\gamma}_N^{(k)}\}_{k\geq 1}$ admits at least one limit point in $\bigoplus_{k\in\mathbb{N}} C([0,T], \mathcal{L}_k^1)$ with respect to the product topology τ_{prod}. Furthermore, all such limit should satisfies a chain of integral equations.

Theorem 3.2. *Let $\tilde{\psi}_N$ be defined as in (3.3.23), $\tilde{\psi}_N(t) = e^{-itH_N}\tilde{\psi}_N(0)$ and $\tilde{\gamma}_N^{(k)}$ be the corresponding k-marginal density. Suppose that $\Gamma_{\infty,t} = \{\tilde{\gamma}^{(k)}\}_{k\geq 1}$ is a limit point of $\tilde{\Gamma}_N = \{\tilde{\gamma}_N^{(k)}\}_{k=1}^N$ in $\bigoplus_{k\in\mathbb{N}} C([0,T], \mathcal{L}_k^1)$ with respect to the product topology τ_{prod}. Then $\Gamma_{\infty,t}$ is a solution to the infinite hierarchy*

$$\gamma^{(k)} = U^{(k)}(t)\gamma^{(k)}(0) - i\sum_{p=1}^{p_0} b_p \sum_{j=1}^{k} \int_0^t ds U^{(k)}(t-s) B_{j;k+1,\dots,k+p}\gamma^{(k+p)}(s), \quad (3.3.32)$$

with initial data $\gamma(0)^{(k)} = |\phi\rangle\langle\phi|^{\otimes k}$. $U^{(k)}(t)$ is the free propagator *defined by $U^{(k)}(t)\gamma^{(k)} := e^{it(\Delta_{\mathbf{x}_k} - \Delta_{\mathbf{x}'_k})}\gamma^{(k)}$.*

We use the following Poincaré type inequality in the proof of Theorem 3.2.

Lemma 3.5 (A Poincaré type inequality)**.** *Let h be a non-negative probability measure on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d}(1+x^2)^{\frac{1}{2}}h(x)dx < \infty$. Then for $h_\epsilon(x) = \frac{1}{\epsilon^d}h(\frac{x}{\epsilon}), \epsilon > 0$, and every $0 \leq \kappa < 1$, there exists a $C > 0$ such that*

$$\begin{aligned}&\big|\mathrm{Tr}J^{(k)}\big(h_\epsilon(x_j - x_{k+1})\cdots h_\epsilon(x_j - x_{k+p}) - \delta(x_j - x_{k+1})\cdots\delta(x_j - x_{k+p})\big)\gamma^{(k+p)}\big| \\ &\qquad \leq C\epsilon^\kappa \big|\!\big|\!\big| J^{(k)} \big|\!\big|\!\big| \mathrm{Tr}\big|S_j S_{k+1}\cdots S_{k+p}\gamma^{(k+p)}S_{k+p}\cdots S_{k+1}S_j\big|,\end{aligned} \tag{3.3.33}$$

for all non-negative $\gamma^{(k+p)} \in \mathcal{L}^1_{k+p}$

Proof. We prove the case $k = 1$ by adapting the arguments in [32, 7]. For the case $k > 1$, the proof is analogous. Since $1 \leq j \leq k$, so $j = 1$ in current case. By the non-negativity of $\gamma^{(1+p)}$, we can decompose it as $\gamma^{(1+p)} = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$, with $\psi_i \in L^2(\mathbb{R}^{(1+p)d})$ and $\lambda_i \geq 0$, $\sum \lambda_i \leq 1$. Then

$$\begin{aligned}&\mathrm{Tr}J^{(1)}\big(h_\epsilon(x_1 - x_2)\cdots h_\epsilon(x_1 - x_{1+p}) - \delta(x_1 - x_2)\cdots\delta(x_1 - x_{1+p})\big)\gamma^{(1+p)} \\ &= \sum_i \lambda_i\langle\psi_i, J^{(1)}\big(h_\epsilon(x_1 - x_2)\cdots h_\epsilon(x_1 - x_{1+p}) - \delta(x_1 - x_2)\cdots\delta(x_1 - x_{1+p})\big)\psi_i\rangle \\ &= \sum_i \lambda_i\langle\Psi_i, \big(h_\epsilon(x_1 - x_2)\cdots h_\epsilon(x_1 - x_{1+p}) - \delta(x_1 - x_2)\cdots\delta(x_1 - x_{1+p})\big)\psi_i\rangle,\end{aligned} \tag{3.3.34}$$

where $\Psi_i = (J^{(1)} \otimes 1)\psi_i$. Next we switch to the Fourier side to obtain

$$\begin{aligned}&\langle\Psi_i, \big(h_\epsilon(x_1 - x_2)\cdots h_\epsilon(x_1 - x_{1+p}) - \delta(x_1 - x_2)\cdots\delta(x_1 - x_{1+p})\big)\psi_i\rangle \\ &= \int dq_1\cdots dq_{1+p}dq'_1\cdots dq'_{1+p}\overline{\hat{\Psi}}_i(q_1, \cdots, q_{1+p})\hat{\psi}_i(q'_1, \cdots, q'_{1+p}) \\ &\quad \times \int dx_2\cdots dx_{1+p}h(x_2)\cdots h(x_{1+p})(e^{i\epsilon x_2(q_2 - q'_2)}\cdots e^{i\epsilon x_{1+p}(q_{1+p} - q'_{1+p})} - 1) \\ &\quad \times \delta(q_1 + \cdots + q_{1+p} - q'_1 - \cdots - q'_{1+p}).\end{aligned} \tag{3.3.35}$$

Since for $x \in \mathbb{R}$, $|e^{ix} - 1| = 2|\sin \frac{x}{2}| \le C|x|^\kappa$ is always true with arbitrary $0 < \kappa < 1$ and constant $C > 0$ independent of κ, we have the following

$$\begin{aligned}\left|e^{i\epsilon x_2(q_2-q_2')} \cdots e^{i\epsilon x_{1+p}(q_{1+p}-q_{1+p}')} - 1\right| &\le C\epsilon^\kappa \Big(\sum_{i=2}^{1+p} |x_i(q_i - q_i')|\Big)^\kappa \\ &\le C\epsilon^\kappa \sum_{i=2}^{1+p} |x_i(q_i - q_i')|^\kappa \\ &\le C\epsilon^\kappa \sum_{i=2}^{1+p} |x_i|^\kappa (|q_i|^\kappa + |q_i'|^\kappa).\end{aligned} \tag{3.3.36}$$

The last inequality follows from $(a+b)^\kappa \le a^\kappa + b^\kappa$ for $\kappa \in (0,1)$ and a, b both nonnegative. And the second to the last inequality follows in a similar way, but with an implicit constant depending on p. Thus

$$\begin{aligned}&\left|\langle \Psi_i, \big(h_\epsilon(x_1 - x_2) \cdots h_\epsilon(x_1 - x_{1+p}) - \delta(x_1 - x_2) \cdots \delta(x_1 - x_{1+p})\big)\psi_i\rangle\right| \\ &\le C\epsilon^\kappa \int dq_1 \cdots dq_{1+p} dq_1' \cdots dq_{1+p}' |\hat{\Psi}_i(q_1, \cdots, q_{1+p})||\hat{\psi}_i(q_1', \cdots, q_{1+p}')| \\ &\times \Big(\prod_{i=2}^{1+p} \int |x_i|^\kappa h(x_i) dx_i\Big)\Big(\sum_{i=2}^{1+p} |q_i|^\kappa + |q_i'|^\kappa\Big)\delta(q_1 + \cdots + q_{1+p} - q_1' - \cdots - q_{1+p}').\end{aligned} \tag{3.3.37}$$

Clearly the p copies of integrations involving h are finite by assumption. And the summation term $\sum_{i=2}^{1+p}(|q_i|^\kappa + |q_i'|^\kappa)$ contains a total of p terms. We will show how to control one of them, say $|q_2|^k$. The final upper bound on this part will be the same (up to a constant p).

$$\begin{aligned}&\int dq_1 \cdots dq_{1+p} dq_1' \cdots dq_{1+p}' \delta(q_1 + \cdots + q_{1+p} - q_1' - \cdots - q_{1+p}') \\ &\times \left|\hat{\Psi}_i(q_1, \cdots, q_{1+p})\right|\left|\hat{\psi}_i(q_1', \cdots, q_{1+p}')\right||q_2|^\kappa \\ &= \int dq_1 \cdots dq_{1+p} dq_1' \cdots dq_{1+p}' \delta(q_1 + \cdots + q_{1+p} - q_1' - \cdots - q_{1+p}')\end{aligned}$$

$$
\begin{aligned}
&\times \frac{\langle q_1\rangle\langle q_2\rangle\cdots\langle q_{1+p}\rangle}{\langle q_1'\rangle\langle q_2'\rangle\cdots\langle q_{1+p}'\rangle}\big|\hat{\Psi}_i(q_1,\cdots,q_{1+p})\big|\frac{\langle q_1'\rangle\langle q_2'\rangle\cdots\langle q_{1+p}'\rangle}{\langle q_1\rangle\langle q_2\rangle^{1-\kappa}\cdots\langle q_{1+p}\rangle}\big|\hat{\psi}_i(q_1',\cdots,q_{1+p}')\big| \\
&\le \rho\int dq_1\cdots dq_{1+p}dq_1'\cdots dq_{1+p}'\delta(q_1+\cdots+q_{1+p}-q_1'-\cdots-q_{1+p}') \\
&\qquad \times \frac{\langle q_1\rangle^2\langle q_2\rangle^2\cdots\langle q_{1+p}\rangle^2}{\langle q_1'\rangle^2\langle q_2'\rangle^2\cdots\langle q_{1+p}'\rangle^2}\big|\hat{\Psi}_i(q_1,\cdots,q_{1+p})\big|^2 \\
&+\frac{1}{\rho}\int dq_1\cdots dq_{1+p}dq_1'\cdots dq_{1+p}'\delta(q_1+\cdots+q_{1+p}-q_1'-\cdots-q_{1+p}') \\
&\qquad\quad \times \frac{\langle q_1'\rangle^2\langle q_2'\rangle^2\cdots\langle q_{1+p}'\rangle^2}{\langle q_1\rangle^2\langle q_2\rangle^{2(1-\kappa)}\langle q_3\rangle^2\cdots\langle q_{1+p}\rangle^2}\big|\hat{\psi}_i(q_1',\cdots,q_{1+p}')\big|^2 \\
&\le \rho\langle\Psi_i,S_1^2S_2^2\cdots S_{1+p}^2\Psi_i\rangle\sup_{Q'}\int\frac{dq_1'\cdots dq_p'}{\langle q_1'\rangle^2\langle q_2'\rangle^2\cdots\langle q_p'\rangle^2\langle Q'-q_1'-\cdots-q_p'\rangle^2} && (3.3.38) \\
&+\frac{1}{\rho}\langle\psi_i,S_1^2S_2^2\cdots S_{1+p}^2\psi_i\rangle\sup_{Q}\int\frac{dq_1dq_3\cdots dq_{1+p}}{\langle q_1\rangle^2\langle Q-q_1-q_3-\cdots-q_p\rangle^{2(1-\kappa)}\langle q_3\rangle^2\cdots\langle q_{1+p}\rangle^2} && (3.3.39)
\end{aligned}
$$

for arbitrary $\rho>0$. We can apply inequality (3.4.2) to the last two integrations (3.3.38) and (3.3.39) for all $\kappa\in(0,1)$ to have

$$
\begin{aligned}
&\big|\mathrm{Tr}J^{(1)}\big(h_\epsilon(x_1-x_2)\cdots h_\epsilon(x_1-x_{1+p})-\delta(x_1-x_2)\cdots\delta(x_1-x_{1+p})\big)\gamma^{(1+p)}\big| \\
&\quad\le C\epsilon^\kappa\big(\rho\mathrm{Tr}J^{(1)}S_1^2S_2^2\cdots S_{1+p}^2J^{(1)}\gamma^{(1+p)}+\frac{1}{\rho}\mathrm{Tr}S_1^2S_2^2\cdots S_{1+p}^2\gamma^{(1+p)}\big) \\
&\quad\le C\epsilon^\kappa\big(\rho\mathrm{Tr}S_1^{-1}J^{(1)}S_1^2J^{(1)}S_1^{-1}S_1S_2\cdots S_{1+p}\gamma^{(1+p)}S_{1+p}\cdots S_2S_1 \\
&\qquad\qquad +\frac{1}{\rho}\mathrm{Tr}S_1^2S_2^2\cdots S_{1+p}^2\gamma^{(1+p)}\big) \\
&\quad\le C\epsilon^\kappa\big(\rho\|S_1^{-1}J^{(1)}S_1\|\|S_1J^{(1)}S_1^{-1}\|+\frac{1}{\rho}\big)\mathrm{Tr}S_1^2S_2^2\cdots S_{1+p}^2\gamma^{(1+p)} \\
&\quad\le C\epsilon^\kappa|||J^{(1)}|||\mathrm{Tr}S_1^2S_2^2\cdots S_{1+p}^2\gamma^{(1+p)}
\end{aligned}
\tag{3.3.40}
$$

by taking $\rho=|||J^{(1)}|||t^{-1}$ in the last inequality.

□

Proof of Theorem 3.2. We adapt the proofs in [7] and [32]. Let $k \geq 1$ be fixed. Up to a subsequence, we can assume that for every $J^{(k)} \in \mathcal{K}_k$

$$\sup_{t\in[0,T]} \mathrm{Tr} J^{(k)}(\gamma^{(k)}(t) - \tilde{\gamma}_N^{(k)}(t)) \to 0, \quad as \;\; N \to \infty. \tag{3.3.41}$$

It is enough to test (3.3.32) for observables in a dense subset of $\mathcal{K}_k$. So we choose an arbitrary $J^{(k)} \in \mathcal{K}_k$ with $\left|\left|\left| J^{(k)} \right|\right|\right| < \infty$. We need to prove

$$\mathrm{Tr} J^{(k)}\gamma^{(k)}(0) = \mathrm{Tr} J^{(k)} \left|\phi\right\rangle \left\langle\phi\right|^{\otimes k}, \tag{3.3.42}$$

and

$$\begin{aligned} &\mathrm{Tr} J^{(k)}\gamma^{(k)} \\ &= \mathrm{Tr} J^{(k)} U^{(k)}(t)\gamma^{(k)}(0) - i\sum_{p=1}^{p_0} b_p \sum_{j=1}^{k} \int_0^t ds \mathrm{Tr} J^{(k)} U^{(k)}(t-s) B_{j;k+1,\dots,k+p}\gamma^{(k+p)}(s). \end{aligned} \tag{3.3.43}$$

By the choice of $J^{(k)}$, (3.3.42) follows from (3.3.41) and (3.3.44):

$$\mathrm{Tr} J^{(k)}(\tilde{\gamma}_N^{(k)}(0) - \left|\phi\right\rangle \left\langle\phi\right|^{\otimes k}) \to 0, \quad as \;\; N \to \infty. \tag{3.3.44}$$

We provide the proof of (3.3.44) in §3.3.3.

For (3.3.43), we use the notation $J_t^{(k)} := J^{(k)} U^{(k)}(t)$, and go back to the BBGKY hierarchy (3.1.5) in the integral form as

$$\mathrm{Tr} J^{(k)}\tilde{\gamma}_N^{(k)}(t) = \tag{3.3.45}$$

$$\mathrm{Tr} J_t^{(k)}\tilde{\gamma}_N^{(k)}(0) \tag{3.3.46}$$

$$-\sum_{p=1}^{p_0} \frac{i}{N^p} \sum_{1\leq i_1<\cdots<i_{p+1}\leq k} \int_0^t ds \mathrm{Tr} J_{t-s}^{(k)}[V_N^{(p)}(x_{i_1}-x_{i_2},\cdots,x_{i_1}-x_{i_{p+1}}), \tilde{\gamma}_N^{(k)}(s)] \tag{3.3.47}$$

$$-\sum_{p=1}^{p_0}\frac{i(N-k)}{N^p}\sum_{1\le i_1<\cdots<i_p\le k}\int_0^t ds\mathrm{Tr}J_{t-s}^{(k)} \tag{3.3.48}$$
$$[V_N^{(p)}(x_{i_1}-x_{i_2},\cdots,x_{i_1}-x_{i_p},x_{i_1}-x_{k+1}),\tilde{\gamma}_N^{(k+1)}(s)]$$
$$-\sum_{p=1}^{p_0}\frac{i(N-k)(N-k-1)}{N^p}\sum_{1\le i_1<\cdots<i_{p-1}\le k}\int_0^t ds\mathrm{Tr}J_{t-s}^{(k)} \tag{3.3.49}$$
$$[V_N^{(p)}(x_{i_1}-x_{i_2},\cdots,x_{i_1}-x_{i_{p-1}},x_{i_1}-x_{k+1},x_{i_2}-x_{k+2}),\tilde{\gamma}_N^{(k+2)}(s)]$$
$$-\cdots$$
$$-\sum_{p=1}^{p_0}\frac{i(N-k)(N-k-1)\cdots(N-k-p+1)}{N^p}\sum_{1\le i_1\le k}\int_0^t ds\mathrm{Tr}J_{t-s}^{(k)} \tag{3.3.50}$$
$$[V_N^{(p)}(x_{i_1}-x_{k+1},x_{i_1}-x_{k+2},\cdots,x_{i_1}-x_{k+p}),\tilde{\gamma}_N^{(k+p)}(s)].$$

Let us look at the behavior of the above terms when $N\to\infty$. Since (3.3.41), (3.3.45) converges to the LHS of (3.3.43); and (3.3.46) converges to the first term on the RHS of (3.3.43). We also observe that all the terms between (3.3.46) and (3.3.50) vanish as $N\to\infty$. Therefore, our goal is to show (3.3.50) converges to the last term on the RHS of (3.3.43). It suffices to prove that for fixed T, k, $J^{(k)}$ and p,

$$\sup_{t\in[s,T]}\Big|\mathrm{Tr}J_{t-s}^{(k)}\big(V_N^{(p)}(x_j-x_{k+1},\cdots,x_j-x_{k+p})\tilde{\gamma}_N^{(k+p)(s)}$$
$$-b_0^{(p)}\delta(x_j-x_{k+1})\cdots\delta(x_j-x_{k+p})\gamma^{(k+p)}(s)\big)\Big|\to 0,\quad as\quad N\to 0. \tag{3.3.51}$$

To bound (3.3.51), we choose a non-negative probability measure h, i.e $h\ge 0$ and $\int h=1$. Define $h_\epsilon(x)=\frac{1}{\epsilon^d}h(\frac{x}{\epsilon}),\epsilon>0$. Then

$$|\mathrm{Tr}J_{t-s}^{(k)}\big(V_N^{(p)}(x_j-x_{k+1},\cdots,x_j-x_{k+p})\tilde{\gamma}_N^{(k+p)}(s)$$
$$-b_0^{(p)}\delta(x_j-x_{k+1})\cdots\delta(x_j-x_{k+p})\gamma^{(k+p)}(s)\big)\Big|$$

$$\leq \left|\mathrm{Tr} J_{t-s}^{(k)} \big(V_N^{(p)}(x_j - x_{k+1}, \cdots, x_j - x_{k+p}) - b_0^{(p)} \delta(x_j - x_{k+1}) \cdots \delta(x_j - x_{k+p}) \big) \tilde{\gamma}_N^{(k+p)}(s) \right| \tag{3.3.52}$$

$$+ b_0^{(p)} \left|\mathrm{Tr} J_{t-s}^{(k)} \big(\delta(x_j - x_{k+1}) \cdots \delta(x_j - x_{k+p}) - h_\epsilon(x_j - x_{k+1}) \cdots h_\epsilon(x_j - x_{k+p}) \big) \tilde{\gamma}_N^{(k+p)}(s) \right| \tag{3.3.53}$$

$$+ b_0^{(p)} \left|\mathrm{Tr} J_{t-s}^{(k)} h_\epsilon(x_j - x_{k+1}) \cdots h_\epsilon(x_j - x_{k+p}) \big(\tilde{\gamma}_N^{(k+p)}(s) - \gamma^{(k+p)}(s) \big) \right| \tag{3.3.54}$$

$$+ b_0^{(p)} \left|\mathrm{Tr} J_{t-s}^{(k)} \big(h_\epsilon(x_j - x_{k+1}) \cdots h_\epsilon(x_j - x_{k+p}) - \delta(x_j - x_{k+1}) \cdots \delta(x_j - x_{k+p}) \big) \gamma^{(k+p)}(s) \right|. \tag{3.3.55}$$

Noting that (note $\int \frac{V^{(p)}}{b_0^{(p)}} = 1$):

- the term (3.3.52) converges to 0 as $N \to \infty$ by Lemma 3.5 and Corollary 3.3;
- the term (3.3.53) converges to 0 uniformly in N as $\epsilon \to 0$ by Lemma 3.5 and Corollary 3.3;
- the term (3.3.54) converges to 0 as $N \to \infty$, for every fixed ϵ (see (6.8) of [32]);
- the term (3.3.55) converges to 0 as $\epsilon \to 0$ by Lemma 3.5 and (3.4.7).

Thus by taking first the limit $N \to \infty$, and then $\epsilon \to 0$, we obtain (3.3.51). $\square$

Proof of Theorem 3.1. By the uniqueness theorems that we will prove in §3.6, we know that for each fixed $\kappa > 0$ and $k \geq 1$, $\hat{\eta}_k(\tilde{\gamma}_N^{(k)}(t), |\phi(t)\rangle \langle \phi(t)|^{\otimes k}) \to 0$

as $N \to \infty$. Or, in other words,

$$\tilde{\gamma}_N^{(k)}(t) \to |\phi(t)\rangle \langle\phi(t)|^{\otimes k} \tag{3.3.56}$$

in the weak* topology of $\mathcal{L}_k^1$. It remains to prove that $\gamma_N^{(k)}(t)$, the k-particle marginal density associated with the original wave functions ψ_N, converges to $|\phi(t)\rangle \langle\phi(t)|^{\otimes k}$ as $N \to \infty$. For any given $\epsilon > 0$, and compact operator $J^{(k)} \in \mathcal{K}_k$, we can find a small enough κ such that (see (3.3.69))

$$\left|\mathrm{Tr} J^{(k)}(\gamma_N^{(k)}(t) - \tilde{\gamma}_N^{(k)}(t))\right| \leq |||J^{(k)}||| \|\psi_N - \tilde{\psi}_N\| < C\kappa^{\frac{1}{2}} \leq \frac{\epsilon}{2} \tag{3.3.57}$$

uniformly in N. With this fixed κ, by (3.3.56), we can pick large enough N to have

$$\left|\mathrm{Tr} J^{(k)}\big(\tilde{\gamma}_N^{(k)}(t) - |\phi(t)\rangle \langle\phi(t)|^{\otimes k}\big)\right| \leq \frac{\epsilon}{2}. \tag{3.3.58}$$

This shows that for any given $\epsilon > 0$ and $J^{(k)} \in \mathcal{K}_k$, $\exists N_0 > 0$ such that

$$\left|\mathrm{Tr} J^{(k)}\big(\gamma_N^{(k)}(t) - |\phi(t)\rangle \langle\phi(t)|^{\otimes k}\big)\right| \leq \epsilon, \tag{3.3.59}$$

whenever $N > N_0$. So for each $t \in [0, T]$ and every k, $\gamma_N^{(k)}(t) \to |\phi(t)\rangle \langle\phi(t)|^{\otimes k}$ in the weak* topology of $\mathcal{L}_k^1$. Since the limiting hierarchy is an orthogonal projection, the convergence in weak* topology is equivalent to the trace norm convergence. This concludes Theorem 3.1. $\square$

3.3.3 Approximation of the initial wave function

Recall the proof of the a priori bound in Corollary 3.3, we need the expectation of H_N^k to be of the order N^k at time 0. The main idea to obtain this is to approximate the initial wave function with cutoffs. We will prove Lemma 3.6 in this section, from which (3.3.44) is immediate.

Lemma 3.6. *Suppose $\psi_N \in L^2(\mathbb{R}^{dN})$ with $\|\psi_N\| = 1$ is a family of N-particle wave functions with the associated marginal densities $\gamma_N^{(k)}$, $k = 1, 2, \cdots$.*
Let χ be a bump function such that $0 \le \chi \le 1$, $\chi(s) = 1$ for $s \in [0,1]$ and $\chi(s) = 0$ for $s \ge 2$. $\kappa > 0$ is a parameter. Define

$$\tilde{\psi}_N := \frac{\chi(\frac{\kappa}{N} H_N)\psi_N}{\left\|\chi(\frac{\kappa}{N} H_N)\psi_N\right\|}. \tag{3.3.60}$$

We denote by $\tilde{\gamma}_N^{(k)}(t)$ the corresponding k-marginal density associated with $\tilde{\psi}_N$. We also assume that

$$\langle \psi_N, H_N \psi_N \rangle \le CN \tag{3.3.61}$$

and

$$\gamma_N^{(1)} \to |\phi\rangle \langle\phi| \quad as \quad N \to \infty \tag{3.3.62}$$

with $\phi \in H^1(\mathbb{R}^d)$. Then, for $\kappa > 0$ small enough and for every $k \le 1$ we have

$$\lim_{N\to\infty} Tr\left|\tilde{\gamma}_N^{(k)} - |\phi\rangle \langle\phi|^{\otimes k}\right| = 0. \tag{3.3.63}$$

Proof. The proof is similar in spirit to the proof for the two-body interactions case, which can be found in [15, 14, 16]. Sketch of the key steps are listed below. We just need to show

$$\mathrm{Tr}\left|\tilde{\gamma}_N^{(1)} - |\phi\rangle \langle\phi|\right| \to 0, \quad as \quad N \to \infty \tag{3.3.64}$$

since (3.3.64) implies (3.3.63) (proved by Lieb and Seiringer in [39]). Moreover, by the equivalence of weak* convergence and trace norm convergence, it is enough to prove that for every compact operator $J^{(1)} \in \mathcal{K}_1$ and for every $\epsilon > 0$, there exists $N_0 = N_0(J^{(1)}, \epsilon)$ such that

$$\left|\mathrm{Tr} J^{(1)}\big(\tilde{\gamma}_N^{(1)} - |\phi\rangle\langle\phi|\big)\right| \leq \epsilon, \quad for \ \ N > N_0. \tag{3.3.65}$$

The proof of (3.3.65) is divided into five steps.

Step 1. By (3.3.62), we know that there exists a sequence $\xi_N^{(N-1)} \in L^2(\mathbb{R}^{d(N-1)})$, $\|\xi_N^{(N-1)}\| = 1$ satisfying

$$\|\psi_N - \phi \otimes \xi_N^{(N-1)}\| \to 0, \quad as \ \ N \to 0. \tag{3.3.66}$$

This was proved by Alessandro Michelangeli in [43]. The proof in the current case can be found in [16].

Step 2. There exists $\phi_* \in H^2(\mathbb{R}^d)$ with $\|\phi_*\| = 1$ such that

$$\|\phi - \phi_*\| \leq \frac{\epsilon}{32|||J^{(1)}|||}. \tag{3.3.67}$$

Step 3. Let $\Xi = \chi(\frac{\kappa}{N} H_N)$. Then by (3.3.61):

$$\begin{aligned}\|(1-\Xi)\psi_N\|^2 &= \langle \psi_N, (1-\Xi)^2 \psi_N\rangle \\ &\leq \langle \psi_N, \mathbf{1}(\kappa H_N \geq N)\psi_N\rangle \\ &\leq \frac{\kappa}{N}\langle \psi_N, H_N \psi_N\rangle \leq C\kappa\end{aligned} \tag{3.3.68}$$

is uniformly in N. Since $\|\psi_N\| = 1$, by triangle inequality we know

$$\|\psi_N - \tilde{\psi}_N\| = \|\frac{\psi_N}{\|\psi_N\|} - \frac{\Xi\psi_N}{\|\Xi\psi_N\|}\| \leq \frac{2}{\|\psi_N\|}\|\psi_N - \Xi\psi_N\| = 2\|(1-\Xi)\psi_N\| \leq C\kappa^{\frac{1}{2}}. \tag{3.3.69}$$

The above inequality is needed in (3.3.57). One can find $\kappa > 0$ small enough such that $\|\Xi\psi_N\| \geq \frac{1}{2}$. Using triangle inequality and noting that $\|\Xi\| \leq 1$, we have

$$
\begin{aligned}
&\Big\|\frac{\Xi\psi_N}{\|\Xi\psi_N\|} - \frac{\Xi(\phi_* \otimes \xi_N^{(N-1)})}{\|\Xi(\phi_* \otimes \xi_N^{(N-1)})\|}\Big\| \\
&= \Big\|\frac{\Xi\psi_N}{\|\Xi\psi_N\|} - \frac{\Xi(\phi_* \otimes \xi_N^{(N-1)})}{\|\Xi\psi_N\|} + \frac{\Xi(\phi_* \otimes \xi_N^{(N-1)})}{\|\Xi\psi_N\|} - \frac{\Xi(\phi_* \otimes \xi_N^{(N-1)})}{\|\Xi(\phi_* \otimes \xi_N^{(N-1)})\|}\Big\| \\
&\leq \frac{1}{\|\Xi\psi_N\|}\big\|\Xi\psi_N - \Xi(\phi_* \otimes \xi_N^{(N-1)})\big\| + \frac{1}{\|\Xi\psi_N\|}\big|\|\Xi\psi_N\| - \|\Xi(\phi_* \otimes \xi_N^{(N-1)})\|\big| \\
&\leq \frac{2}{\|\Xi\psi_N\|}\big\|\Xi\big(\psi_N - \phi_* \otimes \xi_N^{(N-1)}\big)\big\| \\
&\leq 4\big\|\psi_N - \phi_* \otimes \xi_N^{(N-1)}\big\| \\
&\leq 4\big\|\psi_N - \phi \otimes \xi_N^{(N-1)}\big\| + 4\big\|\phi \otimes \xi_N^{(N-1)} - \phi_* \otimes \xi_N^{(N-1)}\big\| \qquad (3.3.70) \\
&\leq 4\big\|\psi_N - \phi \otimes \xi_N^{(N-1)}\big\| + 4\big\|\phi - \phi_*\big\| \\
&\leq \frac{\epsilon}{6|\!|\!|J^{(1)}|\!|\!|} \qquad (3.3.71)
\end{aligned}
$$

for large N. Here in the last inequality we use (3.3.66) and (3.3.67).

Step 4. As in [16] and [7], we define a similar Hamiltonian after taking into account of the $(p+1)$-particle interactions studied in chapter 3.

$$\breve{H}_N := \sum_{i=2}^{N}(-\Delta_{x_i}) + \frac{1}{N^p}\sum_{2\leq i_1<\cdots<i_{p+1}\leq N} V_N^{(p)}\big(x_{i_1}-x_{i_2}, \cdots, x_{i_1}-x_{i_{p+1}}\big). \tag{3.3.72}$$

Instead of acting on all variables, the new Hamiltonian only acts on the last $N-1$ variables. Let $\breve{\Xi} = \chi(\frac{\kappa}{N}\breve{H}_N)$. Then by (3.3.71), we will have

$$\Big\|\frac{\Xi\psi_N}{\|\Xi\psi_N\|} - \frac{\breve{\Xi}(\phi_* \otimes \xi_N^{(N-1)})}{\|\breve{\Xi}(\phi_* \otimes \xi_N^{(N-1)})\|}\Big\| \leq \frac{\epsilon}{3|\!|\!|J^{(1)}|\!|\!|}. \tag{3.3.73}$$

We refer the proof of (3.3.73) to Erdös-Schlein-Yau [14].

Step 5. For (3.3.65), we define

$$\breve{\psi}_N := \frac{\breve{\Xi}(\phi_* \otimes \xi_N^{(N-1)})}{\|\breve{\Xi}(\phi_* \otimes \xi_N^{(N-1)})\|} = \phi_* \otimes \frac{\breve{\Xi}\xi_N^{(N-1)}}{\|\breve{\Xi}\xi_N^{(N-1)}\|} \tag{3.3.74}$$

since $\breve{\Xi}$ only acts on the last $N-1$ variables and $\|\phi_*\| = 1$. Further, we define

$$\breve{\gamma}_N^{(1)}(x_1; x_1') := \int \breve{\psi}_N(x_1; \mathbf{x}_{N-1}) \overline{\breve{\psi}}_N(x_1', \mathbf{x}_{N-1}) d\mathbf{x}_{N-1}. \tag{3.3.75}$$

Note that $\breve{\psi}_N$ is not symmetric in all variables, but it is symmetric in the last $N-1$ variables. Clearly, $\breve{\gamma}_N^{(1)}$ is a density matrix and

$$\breve{\gamma}_N^{(1)} = |\phi_*\rangle \langle\phi_*|\,.$$

Thus, using $\|\tilde{\psi}_N - \breve{\psi}_N\| \leq \frac{\epsilon}{3|||J^{(1)}|||}$, which is equivalent to (3.3.73) and $\|\phi - \phi_*\| \leq \frac{\epsilon}{32|||J^{(1)}|||}$ from (3.3.67), we obtain

$$\begin{aligned}
\left|\mathrm{Tr}J^{(1)}\big(\tilde{\gamma}_N^{(1)} - |\phi\rangle\langle\phi|\big)\right| &\leq \left|\mathrm{Tr}J^{(1)}\big(\tilde{\gamma}_N^{(1)} - \breve{\gamma}_N^{(1)}\big)\right| + \left|\mathrm{Tr}J^{(1)}\big(\,|\phi_*\rangle\langle\phi_*| - |\phi\rangle\langle\phi|\,\big)\right| \\
&\leq 2|||J^{(1)}|||\|\tilde{\psi}_N - \breve{\psi}_N\| + 2|||J^{(1)}|||\|\phi_* - \phi\| \\
&\leq \epsilon
\end{aligned} \tag{3.3.76}$$

for sufficiently large N with arbitrary ϵ and small enough κ. Hence (3.3.65) follows. □

3.4 A priori energy bounds on the limiting hierarchy

This section is a preparation for proving uniqueness theorems in §3.6 using the approach introduced in Klainerman and Machedon [34]. In order to apply [34] we have to establish some energy bounds on the limiting hierarchy. The results are stated in theorems 3.3 and 3.4. From now on, we denote $S^{(k,\alpha)}$ as:

$$S^{(k,\alpha)} = \prod_{j=1}^{k} (1-\Delta_{x_j})^{\frac{\alpha}{2}} (1-\Delta_{x'_j})^{\frac{\alpha}{2}}. \tag{3.4.1}$$

Before we state the theorem about the a priori energy bound, we record an inequality which will be used in the proof of Theorem 3.3.

Lemma 3.7 (An analysis inequality)**.** *Let* $\langle x \rangle = (1+|x|^2)^{\frac{1}{2}}$, $\forall \epsilon_0 \geq 0$, $\alpha \geq 0$ *we have*

$$\int_{\mathbb{R}^d} \frac{dy}{\langle W-y\rangle^{2-2\alpha}\langle y\rangle^2} \leq \begin{cases} \frac{C}{\langle W\rangle^{2-2\alpha}}, & \text{when } \alpha \leq 1 \text{ and } d=1. \\ \frac{C\ln\langle W\rangle}{\langle W\rangle^{2-2\alpha}} \leq \frac{C}{\langle W\rangle^{2-2(\alpha+\epsilon_0)}}, & \text{when } \alpha < 1 \text{ and } d=2. \end{cases} \tag{3.4.2}$$

Proof. We thank Bill Beckner for pointing out an elegant proof of the above estimate. For $d=2$, the inequality follows by spliting $\mathbb{R}^2$ into three pieces, (i) $|y| < \frac{1}{2}|W|$, (ii) $\frac{1}{2}|W| \leq |y| \leq 2|W|$ and (iii) $|y| > 2|W|$. Without loss of generality, we may assume $|W|$ is sufficiently large.

(i). For $|y| < \frac{1}{2}|W|$, we have $|W-y| > \frac{|W|}{2}$ then

$$\begin{aligned} \int_{|y|<\frac{1}{2}|W|} \frac{dy}{\langle W-y\rangle^{2-2\alpha}\langle y\rangle^2} &\leq \frac{C_1}{\langle W\rangle^{2-2\alpha}} \int_{|y|<\frac{1}{2}|W|} \frac{dy}{\langle y\rangle^2} \\ &= \frac{2\pi C_1}{\langle W\rangle^{2-2\alpha}} \int_0^{\frac{1}{2}|W|} \frac{rdr}{1+r^2} \\ &= \frac{C\ln\langle W\rangle}{\langle W\rangle^{2-2\alpha}} \end{aligned} \tag{3.4.3}$$

(ii). For $\frac{1}{2}|W| \le |y| \le 2|W|$, we perform change of variables $W = |W|\theta$, $\theta \in S^1$ and $y = |W|z$.

$$\begin{aligned}
\int_{|y|\in[\frac{1}{2}|W|,2|W|]} \frac{dy}{\langle W-y\rangle^{2-2\alpha}\langle y\rangle^2} &\le \int_{|z|\in[\frac{1}{2},2]} \frac{1}{\langle |W|(\theta-z)\rangle^{2-2\alpha}} \frac{|W|^2}{1+|W|^2|z|^2} dz \\
&\le \int_{|z|\in[\frac{1}{2},2]} \frac{C_1}{\langle |W|(\theta-z)\rangle^{2-2\alpha}} dz \\
&\le \int_{|z|\in[0,3]} \frac{C_1}{\langle |W|z\rangle^{2-2\alpha}} dz \\
&= \int_{r\in[0,3]} \frac{2\pi C_1 r dr}{(1+|W|^2r^2)^{1-\alpha}} \\
&\le \begin{cases} \frac{C_2}{|W|^2}(1+|W|^2)^\alpha \le \frac{C}{\langle W\rangle^{2-2\alpha}}, & \text{for } \alpha > 0. \\ \frac{C_2}{|W|^2}\ln(1+|W|^2) \le \frac{C\ln\langle W\rangle}{\langle W\rangle^2}, & \text{for } \alpha = 0. \end{cases}
\end{aligned} \tag{3.4.4}$$

(iii). For $|y| > 2|W|$, we have $|W-y| > \frac{|y|}{2}$.

$$\begin{aligned}
\int_{|y|>2|W|} \frac{dy}{\langle W-y\rangle^{2-2\alpha}\langle y\rangle^2} &\le \int_{|y|>2|W|} \frac{C_1 dy}{\langle y\rangle^{2-2\alpha}\langle y\rangle^2} \\
&= \int_{|y|>2|W|} \frac{C_1 dy}{(1+|y|^2)^{2-\alpha}} \\
&\le \frac{C}{\langle W\rangle^{2-2\alpha}}
\end{aligned} \tag{3.4.5}$$

Combining (3.4.3) – (3.4.5), we conclude (3.4.2). When $d = 1$, the proof of (3.4.2) is similar.

We remark that when $d = 2$, if we integrate on the domain $|y| < |W|$, we have

$$\int_{|y|<|W|} \frac{dy}{\langle W-y\rangle^{2-2\alpha}\langle y\rangle^2} \ge \int_{|y|<|W|} \frac{dy}{4^{1-\alpha}\langle W\rangle^{2-2\alpha}\langle y\rangle^2} \ge \frac{C\ln\langle W\rangle}{\langle W\rangle^{2-2\alpha}},$$

which implies that the RHS of (3.4.2) is also a lower bound for large $|W|$. Thus for $d = 2$, the integral in (3.4.2) is of order $\mathcal{O}(\frac{\ln\langle W\rangle}{\langle W\rangle^{2-2\alpha}})$. □

Theorem 3.3 (A priori energy bound)**.** *Suppose that $d \in \{1,2\}$, $0 < \beta < \frac{1}{2dp_0+2}$, p satisfies $1 \le p \le p_0$. If $\Gamma_{\infty,t} = \{\gamma^{(k)}(t)\}_{k\ge 1}$ is a limit point of the sequence $\tilde{\Gamma}_{N,t} = \{\tilde{\gamma}_N^{(k)}(t)\}_{k=1}^N$ with respect to the product topology τ_{prod}, then for every $\alpha < 1$ if $d = 2$, and every $\alpha \le 1$ if $d = 1$, there exists $C_\alpha > 0$ (also has $\kappa, p_0, V^{(p)}, d$ dependence) such that*

$$\left\| S^{(k,\alpha)} B_{j;k+1,\cdots,k+p}\gamma^{(k+p)}(t) \right\|_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} \le C_\alpha^{k+p} \tag{3.4.6}$$

for all $k \ge 1$ and all $t \in [0,T]$.

Proof. Since the inequality in Corollary 3.3 is uniformly true for all large N, we can extract an estimate on limit points $\{\gamma^{(k)}(t)\}_{k\ge 1}$ by taking $N \to \infty$:

$$\mathrm{Tr}(1-\Delta_{x_1})\cdots(1-\Delta_{x_k})\gamma^{(k)}(t) \le C^k. \tag{3.4.7}$$

It is enough to prove that

$$\left\| S^{(k,\alpha)} B_{j;k+1,\cdots,k+p}\gamma^{(k+p)}(t) \right\|_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} \le \mathrm{Tr}(1-\Delta_{x_1})\cdots(1-\Delta_{x_{k+p}})\gamma^{(k+p)}(t). \tag{3.4.8}$$

Furthermore, it suffices to show the case that $k = 1$ and $j = 1$ since the proof for other values of k, j is similar. Also, by the definition of the contraction operator $B_{j;k+1,\cdots,k+p}$, we only need to deal with $B^+_{j;k+1,\cdots,k+p}$ (same way works for $B^-_{j;k+1,\cdots,k+p}$). Switching to the Fourier space we have (q_i and q_i' are Fourier conjugate variables of x_i and x_i' respectively):

$$\begin{aligned}&\left(B^+_{1;2,\cdots,1+p}\gamma^{(1+p)}\right)^\wedge(q_1;q_1')\\ &= \int dx_1 dx_1' e^{-ix_1\cdot q_1} e^{ix_1'\cdot q_1'} \int dx_2 dx_2' \cdots dx_{1+p} dx_{1+p}'\end{aligned}$$

$$\times \delta(x_1 - x_2)\delta(x_1 - x_2')\delta(x_1 - x_3)\delta(x_1 - x_3') \cdots \delta(x_1 - x_{1+p})\delta(x_1 - x_{1+p}')$$
$$\times \gamma^{(1+p)}(x_1, \cdots, x_{1+p}; x_1', \cdots, x_{1+p}')$$
$$= \int dq_2 dq_2' \cdots dq_{1+p} dq_{1+p}' \int dx_1 dx_1' \cdots dx_{1+p} dx_{1+p}'$$
$$\times e^{-ix_1\cdot q_1} e^{ix_1'\cdot q_1'} e^{iq_2(x_1-x_2)} e^{-iq_2'(x_1-x_2')} \cdots e^{iq_{1+p}(x_1-x_{1+p})} e^{-iq_{1+p}'(x_1-x_{1+p}')}$$
$$\times \gamma^{(1+p)}(x_1, \cdots, x_{1+p}; x_1', \cdots, x_{1+p}')$$
$$= \int dq_2 dq_2' \cdots dq_{1+p} dq_{1+p}' \int dx_1 dx_1' \cdots dx_{1+p} dx_{1+p}'$$
$$\times e^{-ix_1\cdot(q_1-q_2+q_2'-\cdots-q_{1+p}+q_{1+p}')} e^{-ix_2\cdot q_2} \cdots e^{-ix_{1+p}\cdot q_{1+p}} e^{ix_1'\cdot q_1'} e^{ix_2'\cdot q_2'} \cdots e^{ix_{1+p}' q_{1+p}'}$$
$$\times \gamma^{(1+p)}(x_1, \cdots, x_{1+p}; x_1', \cdots, x_{1+p}')$$
$$= \int dq_2 dq_2' \cdots dq_{1+p} dq_{1+p}'$$
$$\times \hat{\gamma}^{(1+p)}(q_1 - q_2 + q_2' - \cdots - q_{1+p} + q_{1+p}', q_2, \cdots, q_{1+p}; q_1', q_2', \cdots, q_{1+p}').$$

Thus

$$\begin{aligned}
&\left(S^{(1,\alpha)} B^+_{1;2,\cdots,1+p}\gamma^{(1+p)}\right)^\wedge(q_1; q_1') \\
&= \langle q_1\rangle^\alpha \langle q_1'\rangle^\alpha \int dq_2 dq_2' \cdots dq_{1+p} dq_{1+p}' \\
&\qquad \times \hat{\gamma}^{(1+p)}(q_1 - q_2 + q_2' - \cdots - q_{1+p} + q_{1+p}', q_2, \cdots, q_{1+p}; q_1', q_2', \cdots, q_{1+p}'),
\end{aligned} \tag{3.4.9}$$

which implies

$$\begin{aligned}
&\left\|S^{(1,\alpha)} B^+_{1;2,\cdots,1+p}\gamma^{(1+p)}\right\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \\
&= \int dq_1 dq_1' d\tilde{q}_2 d\tilde{\tilde{q}}_2 d\tilde{q}_2' d\tilde{\tilde{q}}_2' \cdots d\tilde{q}_{1+p} d\tilde{\tilde{q}}_{1+p} d\tilde{q}_{1+p}' d\tilde{\tilde{q}}_{1+p}' \langle q_1\rangle^{2\alpha} \langle q_1'\rangle^{2\alpha} \\
&\quad \times \hat{\gamma}^{(1+p)}(q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}', \tilde{q}_2, \cdots, \tilde{q}_{1+p}; \tilde{q}_1', \tilde{q}_2', \cdots, \tilde{q}_{1+p}') \\
&\quad \times \hat{\gamma}^{(1+p)}(q_1 - \tilde{\tilde{q}}_2 + \tilde{\tilde{q}}_2' - \cdots - \tilde{\tilde{q}}_{1+p} + \tilde{\tilde{q}}_{1+p}', \tilde{\tilde{q}}_2, \cdots, \tilde{\tilde{q}}_{1+p}; \tilde{\tilde{q}}_1', \tilde{\tilde{q}}_2', \cdots, \tilde{\tilde{q}}_{1+p}').
\end{aligned} \tag{3.4.10}$$

Note that $\gamma^{(k+p)}$ is non-negative as an operator with trace less than or equal to 1 (see Proposition 3.4). We have the following decomposition

$$
\begin{aligned}
&\hat{\gamma}^{(1+p)}(q_1, q_2, \cdots, q_{1+p}; q_1', q_2', \cdots, q_{1+p}') \\
&= \sum_j \lambda_j \psi_j(q_1, q_2, \cdots, q_{1+p}) \overline{\psi}_j(q_1', q_2', \cdots, q_{1+p}'),
\end{aligned}
\tag{3.4.11}
$$

with $\{\psi_j\}$ an orthonormal system, $\lambda_j \geq 0, \forall j$ and $\sum_j \lambda_j \leq 1$. Applying this decomposition in (3.4.10) yields

$$
\begin{aligned}
&\left\| S^{(1,\alpha)} B^{+}_{1;2,\cdots,1+p} \gamma^{(1+p)} \right\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \\
&= \sum_{i,j} \lambda_i \lambda_j \int dq_1 dq_1' d\tilde{q}_2 d\tilde{\tilde{q}}_2 d\tilde{q}_2' d\tilde{\tilde{q}}_2' \cdots d\tilde{q}_{1+p} d\tilde{\tilde{q}}_{1+p} d\tilde{q}_{1+p}' d\tilde{\tilde{q}}_{1+p}' \langle q_1 \rangle^{2\alpha} \langle q_1' \rangle^{2\alpha} \\
&\quad \times \psi_i(q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}', \tilde{q}_2, \cdots, \tilde{q}_{1+p}) \overline{\psi}_i(\tilde{q}_1', \tilde{q}_2', \cdots, \tilde{q}_{1+p}') \\
&\quad \times \psi_j(q_1 - \tilde{\tilde{q}}_2 + \tilde{\tilde{q}}_2' - \cdots - \tilde{\tilde{q}}_{1+p} + \tilde{\tilde{q}}_{1+p}', \tilde{\tilde{q}}_2, \cdots, \tilde{\tilde{q}}_{1+p}) \overline{\psi}_j(\tilde{\tilde{q}}_1', \tilde{\tilde{q}}_2', \cdots, \tilde{\tilde{q}}_{1+p}').
\end{aligned}
\tag{3.4.12}
$$

Since $\langle x+y \rangle^\alpha \leq 2^\alpha \max\{\langle x \rangle^\alpha, \langle y \rangle^\alpha\} \leq 2^\alpha(\langle x \rangle^\alpha + \langle y \rangle^\alpha)$, we have

$$
\langle q_1 \rangle^\alpha \leq C\Big(\langle q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}' \rangle^\alpha + \langle \tilde{q}_2 \rangle^\alpha + \langle \tilde{q}_2' \rangle^\alpha + \cdots + \langle \tilde{q}_{1+p} \rangle^\alpha + \langle \tilde{q}_{1+p}' \rangle^\alpha \Big)
$$

and

$$
\langle q_1 \rangle^\alpha \leq C\Big(\langle q_1 - \tilde{\tilde{q}}_2 + \tilde{\tilde{q}}_2' - \cdots - \tilde{\tilde{q}}_{1+p} + \tilde{\tilde{q}}_{1+p}' \rangle^\alpha + \langle \tilde{\tilde{q}}_2 \rangle^\alpha + \langle \tilde{\tilde{q}}_2' \rangle^\alpha + \cdots + \langle \tilde{\tilde{q}}_{1+p} \rangle^\alpha + \langle \tilde{\tilde{q}}_{1+p}' \rangle^\alpha \Big).
$$

Multiplying them together we have the following estimate:

$$
\begin{aligned}
\langle q_1 \rangle^{2\alpha} \leq\ & C\Big(\langle q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}' \rangle^\alpha + \langle \tilde{q}_2 \rangle^\alpha + \langle \tilde{q}_2' \rangle^\alpha + \cdots + \langle \tilde{q}_{1+p} \rangle^\alpha + \langle \tilde{q}_{1+p}' \rangle^\alpha \Big) \\
& \times \Big(\langle q_1 - \tilde{\tilde{q}}_2 + \tilde{\tilde{q}}_2' - \cdots - \tilde{\tilde{q}}_{1+p} + \tilde{\tilde{q}}_{1+p}' \rangle^\alpha + \langle \tilde{\tilde{q}}_2 \rangle^\alpha + \langle \tilde{\tilde{q}}_2' \rangle^\alpha + \cdots + \langle \tilde{\tilde{q}}_{1+p} \rangle^\alpha + \langle \tilde{\tilde{q}}_{1+p}' \rangle^\alpha \Big).
\end{aligned}
\tag{3.4.13}
$$

After substituting the above bound in (3.4.12), we will obtain $(2p+1)^2$ contributed terms. However, it is enough to illustrate how to control just one of them, since the remaining cases are essentially the same. For instance, the first contribution comes from the replacement of the factor $\langle q_1\rangle^{2\alpha}$ on the RHS of (3.4.12) by $\langle q_1-\tilde{q}_2+\tilde{q}'_2-\cdots-\tilde{q}_{1+p}+\tilde{q}'_{1+p}\rangle^{\alpha}\langle q_1-\tilde{\tilde{q}}_2+\tilde{\tilde{q}}'_2-\cdots-\tilde{\tilde{q}}_{1+p}+\tilde{\tilde{q}}'_{1+p}\rangle^{\alpha}$. Using Schwartz inequality we find

$$
\begin{aligned}
&\int dq_1 dq'_1 d\tilde{q}_2 d\tilde{\tilde{q}}_2 d\tilde{q}'_2 d\tilde{\tilde{q}}'_2 \cdots d\tilde{q}_{1+p} d\tilde{\tilde{q}}_{1+p} d\tilde{q}'_{1+p} d\tilde{\tilde{q}}'_{1+p} \\
&\quad\times \langle q_1-\tilde{q}_2+\tilde{q}'_2-\cdots-\tilde{q}_{1+p}+\tilde{q}'_{1+p}\rangle^{\alpha}\langle q_1-\tilde{\tilde{q}}_2+\tilde{\tilde{q}}'_2-\cdots-\tilde{\tilde{q}}_{1+p}+\tilde{\tilde{q}}'_{1+p}\rangle^{\alpha}\langle q'_1\rangle^{2\alpha} \\
&\quad\times \psi_i(q_1-\tilde{q}_2+\tilde{q}'_2-\cdots-\tilde{q}_{1+p}+\tilde{q}'_{1+p},\tilde{q}_2,\cdots,\tilde{q}_{1+p})\overline{\psi}_i(\tilde{q}'_1,\tilde{q}'_2,\cdots,\tilde{q}'_{1+p}) \\
&\quad\times \psi_j(q_1-\tilde{\tilde{q}}_2+\tilde{\tilde{q}}'_2-\cdots-\tilde{\tilde{q}}_{1+p}+\tilde{\tilde{q}}'_{1+p},\tilde{\tilde{q}}_2,\cdots,\tilde{\tilde{q}}_{1+p})\overline{\psi}_j(\tilde{\tilde{q}}'_1,\tilde{\tilde{q}}'_2,\cdots,\tilde{\tilde{q}}'_{1+p}) \\
&\le A+B,
\end{aligned}
\tag{3.4.14}
$$

where

$$
\begin{aligned}
A=&\int dq_1 dq'_1 d\tilde{q}_2 d\tilde{\tilde{q}}_2 d\tilde{q}'_2 d\tilde{\tilde{q}}'_2 \cdots d\tilde{q}_{1+\frac{p}{2}} d\tilde{\tilde{q}}_{1+p} d\tilde{q}'_{1+p} d\tilde{\tilde{q}}'_{1+p}\langle q'_1\rangle^{2\alpha} \\
&\times\frac{\langle q_1-\tilde{q}_2+\tilde{q}'_2-\cdots-\tilde{q}_{1+p}+\tilde{q}'_{1+p}\rangle^{2}\langle\tilde{q}_2\rangle^2\langle\tilde{q}_3\rangle^2\cdots\langle\tilde{q}_{1+p}\rangle^2\langle\tilde{\tilde{q}}'_2\rangle^2\langle\tilde{\tilde{q}}'_3\rangle^2\cdots\langle\tilde{\tilde{q}}'_{1+p}\rangle^2}{\langle q_1-\tilde{\tilde{q}}_2+\tilde{\tilde{q}}'_2-\cdots-\tilde{\tilde{q}}_{1+p}+\tilde{\tilde{q}}'_{1+p}\rangle^{2-2\alpha}\langle\tilde{\tilde{q}}_2\rangle^2\langle\tilde{\tilde{q}}_3\rangle^2\cdots\langle\tilde{\tilde{q}}_{1+p}\rangle^2\langle\tilde{q}'_2\rangle^2\langle\tilde{q}'_3\rangle^2\cdots\langle\tilde{q}'_{1+p}\rangle^2} \\
&\times\left|\psi_i(q_1-\tilde{q}_2+\tilde{q}'_2-\cdots-\tilde{q}_{1+p}+\tilde{q}'_{1+p},\tilde{q}_2,\cdots,\tilde{q}_{1+p})\right|^2\left|\psi_j(\tilde{\tilde{q}}'_1,\tilde{\tilde{q}}'_2,\cdots,\tilde{\tilde{q}}'_{1+p})\right|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
B=&\int dq_1 dq'_1 d\tilde{q}_2 d\tilde{\tilde{q}}_2 d\tilde{q}'_2 d\tilde{\tilde{q}}'_2 \cdots d\tilde{q}_{1+\frac{p}{2}} d\tilde{\tilde{q}}_{1+p} d\tilde{q}'_{1+p} d\tilde{\tilde{q}}'_{1+p}\langle q'_1\rangle^{2\alpha} \\
&\times\frac{\langle q_1-\tilde{\tilde{q}}_2+\tilde{\tilde{q}}'_2-\cdots-\tilde{\tilde{q}}_{1+p}+\tilde{\tilde{q}}'_{1+p}\rangle^{2}\langle\tilde{\tilde{q}}_2\rangle^2\langle\tilde{\tilde{q}}_3\rangle^2\cdots\langle\tilde{\tilde{q}}_{1+p}\rangle^2\langle\tilde{q}'_2\rangle^2\langle\tilde{q}'_3\rangle^2\cdots\langle\tilde{q}'_{1+p}\rangle^2}{\langle q_1-\tilde{q}_2+\tilde{q}'_2-\cdots-\tilde{q}_{1+p}+\tilde{q}'_{1+p}\rangle^{2-2\alpha}\langle\tilde{q}_2\rangle^2\langle\tilde{q}_3\rangle^2\cdots\langle\tilde{q}_{1+p}\rangle^2\langle\tilde{\tilde{q}}'_2\rangle^2\langle\tilde{\tilde{q}}'_3\rangle^2\cdots\langle\tilde{\tilde{q}}'_{1+p}\rangle^2} \\
&\times\left|\psi_j(q_1-\tilde{\tilde{q}}_2+\tilde{\tilde{q}}'_2-\cdots-\tilde{\tilde{q}}_{1+p}+\tilde{\tilde{q}}'_{1+p},\tilde{\tilde{q}}_2,\cdots,\tilde{\tilde{q}}_{1+p})\right|^2\left|\psi_i(\tilde{q}'_1,\tilde{q}'_2,\cdots,\tilde{q}'_{1+p})\right|^2.
\end{aligned}
$$

Now let us focus on A, as B can be handled similarly. Performing integration on $\tilde{\tilde{q}}_2, \tilde{\tilde{q}}_3, \cdots, \tilde{\tilde{q}}_{1+p}$ we obtain:

$$\begin{aligned}A \leq C &\int dq_1 dq_1' d\tilde{q}_2 d\tilde{q}_2' d\tilde{\tilde{q}}_2' \cdots d\tilde{q}_{1+p} d\tilde{q}_{1+p}' d\tilde{\tilde{q}}_{1+p}' \langle q_1' \rangle^{2\alpha} \\ &\times \frac{\langle q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}' \rangle^2 \langle \tilde{q}_2 \rangle^2 \langle \tilde{q}_3 \rangle^2 \cdots \langle \tilde{q}_{1+p} \rangle^2 \langle \tilde{\tilde{q}}_2' \rangle^2 \langle \tilde{\tilde{q}}_3' \rangle^2 \cdots \langle \tilde{\tilde{q}}_{1+p}' \rangle^2}{\langle q_1 + \tilde{\tilde{q}}_2' + \cdots + \tilde{\tilde{q}}_{1+p}' \rangle^{2-2(\alpha+(p-1)\epsilon_0)} \langle \tilde{q}_2' \rangle^2 \langle \tilde{q}_3' \rangle^2 \cdots \langle \tilde{q}_{1+p}' \rangle^2} \\ &\times \left| \psi_i(q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}', \tilde{q}_2, \cdots, \tilde{q}_{1+p}) \right|^2 \left| \psi_j(\tilde{\tilde{q}}_1', \tilde{\tilde{q}}_2', \cdots, \tilde{\tilde{q}}_{1+p}') \right|^2,\end{aligned} \tag{3.4.15}$$

where we repeatedly used Lemma 3.7 $p-1$ times. When $d = 2$, we choose ϵ_0 such that $\alpha + (p-1)\epsilon_0 < 1$, for instance, $\epsilon_0 = \frac{1-\alpha}{2(p-1)}$. Actually, the case $d = 1$ can be merged to the case $d = 2$ by setting $\epsilon_0 = 0$.

Let $\breve{q}_1 = q_1 - \tilde{q}_2 + \tilde{q}_2' - \cdots - \tilde{q}_{1+p} + \tilde{q}_{1+p}'$ in (3.4.15). Since $\alpha \leq 1$, we can replace $\langle q_1' \rangle^{2\alpha}$ with $\langle q_1' \rangle^2$ for an upper bound:

$$\begin{aligned}A \leq C &\int d\breve{q}_1 dq_1' d\tilde{q}_2 d\tilde{q}_2' d\tilde{\tilde{q}}_2' \cdots d\tilde{q}_{1+p} d\tilde{q}_{1+p}' d\tilde{\tilde{q}}_{1+p}' \\ &\times \frac{\langle \breve{q}_1 \rangle^2 \langle \tilde{q}_2 \rangle^2 \langle \tilde{q}_3 \rangle^2 \cdots \langle \tilde{q}_{1+p} \rangle^2 \langle q_1' \rangle^2 \langle \tilde{\tilde{q}}_2' \rangle^2 \langle \tilde{\tilde{q}}_3' \rangle^2 \cdots \langle \tilde{\tilde{q}}_{1+p}' \rangle^2}{\langle \breve{q}_1 + \tilde{q}_2 - \tilde{q}_2' + \cdots + \tilde{q}_{1+p} - \tilde{q}_{1+p}' + \tilde{\tilde{q}}_2' + \cdots + \tilde{\tilde{q}}_{1+p}' \rangle^{2-2(\alpha+(p-1)\epsilon_0)} \langle \tilde{q}_2' \rangle^2 \langle \tilde{q}_3' \rangle^2 \cdots \langle \tilde{q}_{1+p}' \rangle^2} \\ &\times \left| \psi_i(\breve{q}_1, \tilde{q}_2, \cdots, \tilde{q}_{1+p}) \right|^2 \left| \psi_j(\tilde{\tilde{q}}_1', \tilde{\tilde{q}}_2', \cdots, \tilde{\tilde{q}}_{1+p}') \right|^2 \\ \leq CC_\alpha' &\int d\breve{q}_1 d\tilde{q}_2 d\tilde{q}_3 \cdots d\tilde{q}_{1+p} \langle \breve{q}_1 \rangle^2 \langle \tilde{q}_2 \rangle^2 \langle \tilde{q}_3 \rangle^2 \cdots \langle \tilde{q}_{1+p} \rangle^2 \left| \psi_i(\breve{q}_1, \tilde{q}_2, \cdots, \tilde{q}_{1+p}) \right|^2 \\ &\times \int dq_1' d\tilde{\tilde{q}}_2' d\tilde{\tilde{q}}_3' \cdots d\tilde{\tilde{q}}_{1+p}' \langle q_1' \rangle^2 \langle \tilde{\tilde{q}}_2' \rangle^2 \langle \tilde{\tilde{q}}_3' \rangle^2 \cdots \langle \tilde{\tilde{q}}_{1+p}' \rangle^2 \left| \psi_j(\tilde{\tilde{q}}_1', \tilde{\tilde{q}}_2', \cdots, \tilde{\tilde{q}}_{1+p}') \right|^2,\end{aligned} \tag{3.4.16}$$

where C_α' is defined as

$$C_\alpha' = \sup_{W \in \mathbb{R}^d} \int \frac{dx_1 dx_2 \cdots dx_p}{\langle W - x_1 - x_2 - \cdots - x_p \rangle^{2-2(\alpha+(p-1)\epsilon_0)} \langle x_1 \rangle^2 \langle x_2 \rangle^2 \cdots \langle x_p \rangle^2}. \tag{3.4.17}$$

For all $\alpha \leq 1$ if $d = 1$ and $\alpha < 1$ if $d = 2$, $C'_\alpha < \infty$. Now we have the control on one of the $(2p+1)^2$ pieces, and the remaining pieces can be bounded the same way. Recalling (3.4.11) and (3.4.12), we can conclude that

$$
\begin{aligned}
&\big\| S^{(1,\alpha)} B^{+}_{1;2,\cdots,1+p} \gamma^{(1+p)} \big\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \\
&\leq C_\alpha \sum_{i,j} \lambda_i \lambda_j \int d\breve{q}_1 d\tilde{q}_2 d\tilde{q}_3 \cdots d\tilde{q}_{1+p} \langle \breve{q}_1 \rangle^2 \langle \tilde{q}_2 \rangle^2 \langle \tilde{q}_3 \rangle^2 \cdots \langle \tilde{q}_{1+p} \rangle^2 \big| \psi_i(\breve{q}_1, \tilde{q}_2, \cdots, \tilde{q}_{1+p}) \big|^2 \\
&\qquad \times \int dq'_1 d\tilde{q}'_2 d\tilde{q}'_3 \cdots d\tilde{q}'_{1+p} \langle q'_1 \rangle^2 \langle \tilde{q}'_2 \rangle^2 \langle \tilde{q}'_3 \rangle^2 \cdots \langle \tilde{q}'_{1+p} \rangle^2 \big| \psi_j(\tilde{q}'_1, \tilde{q}'_2, \cdots, \tilde{q}'_{1+p}) \big|^2 \\
&\leq C_\alpha \Big(\int d\breve{q}_1 d\tilde{q}_2 d\tilde{q}_3 \cdots d\tilde{q}_{1+p} \langle \breve{q}_1 \rangle^2 \langle \tilde{q}_2 \rangle^2 \langle \tilde{q}_3 \rangle^2 \cdots \langle \tilde{q}_{1+p} \rangle^2 \\
&\qquad\qquad \times \big| \hat{\gamma}^{(1+p)}(\breve{q}_1, \tilde{q}_2, \cdots, \tilde{q}_{1+p}; \breve{q}_1, \tilde{q}_2, \cdots, \tilde{q}_{1+p}) \big|^2 \Big)^2 \\
&= C_\alpha \Big(\mathrm{Tr}(1 - \Delta_{x_1}) \cdots (1 - \Delta_{x_{k+p}}) \gamma^{(k+p)} \Big)^2.
\end{aligned}
$$

Therefore, (3.4.8) follows.

□

Theorem 3.4 (A priori induction estimate)**.** *Suppose that $d \geq 1$. If $\Gamma_{\infty,t} = \{\gamma^{(k)}\}_{k \geq 1}$ is a limit point of the sequence $\tilde{\Gamma}_{N,t} = \{\tilde{\gamma}^{(k)}_N(t)\}_{k=1}^N$ with respect to the product topology τ_{prod}, then, for every $\alpha > \frac{d}{2}$ there exists a constant C_α (also depends on p_0, d) such that the estimate*

$$
\Big\| S^{(k,\alpha)} B_{j;k+1,\cdots,k+p} \gamma^{(k+p)} \Big\|_{L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk})} \leq C_\alpha \Big\| S^{(k+p,\alpha)} \gamma^{(k+p)} \Big\|_{L^2\left(\mathbb{R}^{d(k+p)} \times \mathbb{R}^{d(k+p)}\right)} \tag{3.4.18}
$$

holds.

Proof. We will work on the Fourier side of spacial coordinates. Let $(\mathbf{u}_k, \mathbf{u}'_k)$, $\mathbf{q} := (q_1, q_2, \cdots, q_p)$ and $\mathbf{q}' := (q'_1, q'_2, \cdots, q'_p)$ be the Fourier conjugate variables corresponding to $(\mathbf{x}_k, \mathbf{x}'_k)$, $(x_{k+1}, x_{k+2}, \cdots, x_{k+p})$ and $(x'_{k+1}, x'_{k+2}, \cdots, x'_{k+p})$ respectively.

Assume, without loss of generality, $j = 1$ in $B_{j;k+1,\cdots,k+p}$. We replace the contraction operator by its positive part $B^+_{j;k+1,\cdots,k+p}$ here, since the negative part is similar. By Plancherel's theorem

$$
\begin{aligned}
&\left\| S^{(k,\alpha)} B_{1;k+1,\cdots,k+p} \gamma^{(k+p)} \right\|^2_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} \\
&= \int d\mathbf{u}_k d\mathbf{u}'_k \prod_{j=1}^{k} \langle u_j \rangle^{2\alpha} \langle u'_j \rangle^{2\alpha} \\
&\quad \times \Big(\int d\mathbf{q} d\mathbf{q}' \hat{\gamma}^{(k+p)}(u_1 + q_1 + \cdots + q_p - q'_1 - \cdots - q'_p, u_2, \cdots, u_k, \mathbf{q}; \mathbf{u}'_k, \mathbf{q}') \Big)^2.
\end{aligned}
\tag{3.4.19}
$$

Cauchy-Schwartz inequality gives us an upper bound

$$
\begin{aligned}
(3.4.19) \leq &\int d\mathbf{u}_k d\mathbf{u}'_k F_\alpha(\mathbf{u}_k, \mathbf{u}'_k) \prod_{j=2}^{k} \langle u_j \rangle^{2\alpha} \prod_{j=1}^{k} \langle u'_j \rangle^{2\alpha} \int d\mathbf{q} d\mathbf{q}' \\
&\times \langle u_1 + q_1 + \cdots + q_p - q'_1 - \cdots - q'_p \rangle^{2\alpha} \langle q_1 \rangle^{2\alpha} \cdots \langle q_p \rangle^{2\alpha} \langle q'_1 \rangle^{2\alpha} \cdots \langle q'_p \rangle^{2\alpha} \\
&\times |\hat{\gamma}^{(k+p)}(u_1 + q_1 + \cdots + q_p - q'_1 - \cdots - q'_p, u_2, \cdots, u_k, \mathbf{q}; \mathbf{u}'_k, \mathbf{q}')|^2 \\
\leq &\sup_{\mathbf{u}_k, \mathbf{u}'_k} F_\alpha(\mathbf{u}_k, \mathbf{u}'_k) \times \left\| S^{(k+p,\alpha)} \gamma^{(k+p)} \right\|_{L^2(\mathbb{R}^{d(k+p)}\times\mathbb{R}^{d(k+p)})},
\end{aligned}
\tag{3.4.20}
$$

where

$$
\begin{aligned}
&F_\alpha(\mathbf{u}_k, \mathbf{u}'_k) := \\
&\quad \int \frac{\langle u_1 \rangle^{2\alpha} d\mathbf{q} d\mathbf{q}'}{\langle u_1 + q_1 + \cdots + q_p - q'_1 - \cdots - q'_p \rangle^{2\alpha} \langle q_1 \rangle^{2\alpha} \cdots \langle q_p \rangle^{2\alpha} \langle q'_1 \rangle^{2\alpha} \cdots \langle q'_p \rangle^{2\alpha}}.
\end{aligned}
\tag{3.4.21}
$$

Because of our simplifications at the beginning (specifying j and neglecting the negative part of $B_{j;k+1,\cdots,k+p}$), the function $F_\alpha(\mathbf{u}_k, \mathbf{u}'_k)$ only depends on u_1.

Using

$$\langle u_1\rangle^{2\alpha} \le C\big(\langle u_1 + q_1 + \cdots + q_p - q_1' - \cdots - q_p'\rangle^{2\alpha} + \langle q_1\rangle^{2\alpha} + \cdots + \langle q_p\rangle^{2\alpha}$$
$$+ \langle q_1'\rangle^{2\alpha} + \cdots + \langle q_p'\rangle^{2\alpha}\big),$$

we shift some of the momentum variables to obtain

$$\sup_{\mathbf{u}_k, \mathbf{u}_k'} F_\alpha(\mathbf{u}_k, \mathbf{u}_k') \le C \int \frac{d\mathbf{q}d\mathbf{q}'}{\langle q_1\rangle^{2\alpha} \cdots \langle q_p\rangle^{2\alpha} \langle q_1'\rangle^{2\alpha} \cdots \langle q_p'\rangle^{2\alpha}}. \tag{3.4.22}$$

The RHS of (3.4.22) is always finite when $\alpha > \frac{d}{2}$. This proves the theorem. $\square$

The above estimate (3.4.18) requires that $\alpha > \frac{d}{2}$. Recall the conditions on α ($\alpha < 1$ if $d > 2$, $\alpha \le 1$ if $d = 1$) in Theorem 3.3. If we want to use both theorems, only $d = 1$ gives us a nonempty intersection of the two conditions, so we cannot afford this when $d > 1$. However we need a bound like (3.4.18) for iterative computations in the proof of uniqueness of the limiting hierarchy. We build such a bound in the next section.

3.5 Bounds on the free evolution

In this section, we consider the case when the interactions among particles are neglected (i.e., $b_p = 0$). We will prove a Strichartz type estimate that can be used when dealing with recursive Duhamel expansion terms. The approach we followed in this part is exhibited in [34, 32, 7].

Theorem 3.5 (Free evolving bound)**.** *For fixed $p_0 \ge 1$, assume that $d = 2$, $1 - \frac{1}{2(2p_0-1)} < \alpha < 1$, $1 \le p \le p_0$. Let $\gamma^{(k+p)}$ denote the solution of*

$$i\partial_t \gamma^{(k+p)}(t, \mathbf{x}_{k+p}, \mathbf{x}_{k+p}') + (\Delta_{\mathbf{x}_{k+p}} - \Delta_{\mathbf{x}_{k+p}'})\gamma^{(k+p)}(t, \mathbf{x}_{k+p}, \mathbf{x}_{k+p}') = 0, \tag{3.5.1}$$

with initial condition

$$\gamma^{(k+p)}(0,\cdot)=\gamma_0^{(k+p)}\in\mathcal{H}^\alpha, \tag{3.5.2}$$

where $\mathcal{H}^\alpha$ denotes the space of density matrices with finite Hilbert-Schmidt type Sobolev norms:

$$\mathcal{H}^\alpha=\{\gamma^{(k)}:\|S^{(k,\alpha)}\gamma^{(k)}\|_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})}<\infty\}. \tag{3.5.3}$$

Then, there exists a constant $C=C_\alpha$ (also depends on p_0) but independent of j,k such that

$$\begin{aligned}&\left\|S^{(k,\alpha)}B_{j;k+1,\cdots,k+p}\gamma^{(k+p)}\right\|_{L^2_{t,\mathbf{x}_k,\mathbf{x}'_k}(\mathbb{R}\times\mathbb{R}^{2k}\times\mathbb{R}^{2k})}\\ &\leq C_\alpha\left\|S^{(k+p,\alpha)}\gamma_0^{(k+p)}\right\|_{L^2_{\mathbf{x}_{k+p},\mathbf{x}'_{k+p}}(\mathbb{R}^{2(k+p)}\times\mathbb{R}^{2(k+p)})}\end{aligned} \tag{3.5.4}$$

holds.

Proof. Following [7], since the two norms are both L^2 norms, by Plancherel's theorem, it suffices to prove the estimate (3.5.4) for the Fourier transform of functions on both sides. As before, by definition of the contraction operator in (3.1.9) and (3.1.10), we only need to estimate the term in $B^+_{j;k+1,\cdots,k+p}$; the term in $B^-_{j;k+1,\cdots,k+p}$ can be treated in the same manner. Let $(\tau,\mathbf{u}_k,\mathbf{u}'_k)$, $\mathbf{q}:=(q_1,q_2,\cdots,q_p)$ and $\mathbf{q}':=(q'_1,q'_2,\cdots,q'_p)$ be the Fourier conjugate variables corresponding to $(t,\mathbf{x}_k,\mathbf{x}'_k)$, $(x_{k+1},x_{k+2},\cdots,x_{k+p})$ and $(x'_{k+1},x'_{k+2},\cdots,x'_{k+p})$ respectively. For convenience, let

$$\delta(\cdots):=\delta(\tau+(u_1+q_1+q_2+\cdots+q_p-q'_1-q'_2-\cdots-q'_p)^2+\sum_{j=2}^k u_j^2+|\mathbf{q}|^2-|\mathbf{u}'_k|^2-|\mathbf{q}'|^2). \tag{3.5.5}$$

We may also assume that $j=1$ in $B_{j;k+1,\cdots,k+p}$ without loss of generality. Then

$$\begin{aligned}
&\left\|S^{(k,\alpha)}B_{1;k+1,\cdots,k+p}\gamma^{(k+p)}\right\|^2_{L^2_{t,\mathbf{x}_k,\mathbf{x}'_k}(\mathbb{R}\times\mathbb{R}^{2k}\times\mathbb{R}^{2k})}\\
&=\int_{\mathbb{R}}d\tau\int d\mathbf{u}_k d\mathbf{u}'_k\prod_{j=1}^{k}\langle u_j\rangle^{2\alpha}\langle u'_j\rangle^{2\alpha}\Big(\int d\mathbf{q}d\mathbf{q}'\delta(\cdots)\\
&\qquad\times\hat{\gamma}^{(k+p)}(\tau,u_1+q_1+\cdots+q_p-q'_1-\cdots-q'_p,u_2,\cdots,u_k,\mathbf{q},\mathbf{u}'_k,\mathbf{q}')\Big)^2.
\end{aligned}\tag{3.5.6}$$

Applying the Cauchy-Schwarz inequality, the above integral is further bounded by:

$$\begin{aligned}
(3.5.6)\leq&\int_{\mathbb{R}}d\tau\int d\mathbf{u}_k d\mathbf{u}'_k I_{\alpha,p}(\tau,\mathbf{u}_k,\mathbf{u}'_k)\prod_{j=2}^{k}\langle u_j\rangle^{2\alpha}\prod_{j=1}^{k}\langle u'_j\rangle^{2\alpha}\int d\mathbf{q}d\mathbf{q}'\delta(\cdots)\\
&\times\langle u_1+q_1+\cdots+q_p-q'_1-\cdots-q'_p\rangle^{2\alpha}\langle q_1\rangle^{2\alpha}\cdots\langle q_p\rangle^{2\alpha}\langle q'_1\rangle^{2\alpha}\cdots\langle q'_p\rangle^{2\alpha}\\
&\times|\hat{\gamma}^{(k+p)}(\tau,u_1+q_1+\cdots+q_p-q'_1-\cdots-q'_p,u_2,\cdots,u_k,\mathbf{q},\mathbf{u}'_k,\mathbf{q}')|^2\\
\leq&\left\|S^{(k+p,\alpha)}\gamma_0^{(k+p)}\right\|_{L^2_{\mathbf{x}_{k+p},\mathbf{x}'_{k+p}}(\mathbb{R}^{2(k+p)}\times\mathbb{R}^{2(k+p)})}\times\sup_{\tau,\mathbf{u}_k,\mathbf{u}'_k}I_{\alpha,p}(\tau,\mathbf{u}_k,\mathbf{u}'_k),
\end{aligned}$$

where

$$\begin{aligned}
&I_{\alpha,p}(\tau,\mathbf{u}_k,\mathbf{u}'_k):=\\
&\int d\mathbf{q}d\mathbf{q}'\frac{\delta(\cdots)\langle u_1\rangle^{2\alpha}}{\langle u_1+q_1+\cdots+q_p-q'_1-\cdots-q'_p\rangle^{2\alpha}\langle q_1\rangle^{2\alpha}\cdots\langle q_p\rangle^{2\alpha}\langle q'_1\rangle^{2\alpha}\cdots\langle q'_p\rangle^{2\alpha}}.
\end{aligned}\tag{3.5.7}$$

If we can show that the supremum of $I_{\alpha,p}$ over $\tau,\mathbf{u}_k,\mathbf{u}'_k$ is bounded by a constant (which only depends on α) then we are done. Now, observe that

$$\begin{aligned}
\langle u_1\rangle^{2\alpha}\leq C\big(&\langle u_1+q_1+\cdots+q_p-q'_1-\cdots-q'_p\rangle^{2\alpha}+\langle q_1\rangle^{2\alpha}+\cdots+\langle q_p\rangle^{2\alpha}\\
&+\langle q'_1\rangle^{2\alpha}+\cdots+\langle q'_p\rangle^{2\alpha}\big).
\end{aligned}\tag{3.5.8}$$

So we have the following:

$$I_{\alpha,p}(\tau, \mathbf{u}_k, \mathbf{u}'_k) \le \sum_{l=1}^{2p+1} J_l, \tag{3.5.9}$$

where J_l is obtained by using (3.5.8) and canceling the corresponding term in the denominator of (3.5.7). For example,

$$J_1 \le C \int d\mathbf{q}d\mathbf{q}' \frac{\delta(\cdots)}{\langle q_1\rangle^{2\alpha}\cdots\langle q_p\rangle^{2\alpha}\langle q'_1\rangle^{2\alpha}\cdots\langle q'_p\rangle^{2\alpha}} \tag{3.5.10}$$

and each J_l for $l = 2, 3, \cdots, 2p+1$ can be brought into a similar form by appropriately translating one of the momenta q_j, q'_j. Following [34, 32, 7], we observe the argument of the δ distribution equals to

$$\begin{aligned} Arg[\delta] = \tau + (u_1 + q_1 + \cdots + q_p - q'_1 - \cdots - q'_{p-1})^2 + \sum_{j=2}^{k} u_j^2 + |\mathbf{q}|^2 \\ - |\mathbf{u}'_k|^2 - |\mathbf{q}'|^2 + (q'_p)^2 - 2(u_1 + q_1 + \cdots + q_p - q'_1 - \cdots - q'_{p-1}) \cdot q'_p. \end{aligned}$$

Then we integrate out the δ distribution using the component of q'_p parallel to $u_1 + q_1 + \cdots + q_p - q'_1 - \cdots - q'_{p-1}$, which yields

$$J_1 \le \int \frac{C_\alpha C d\mathbf{q} dq'_1 \cdots dq'_{p-1}}{|u_1 + q_1 + \cdots + q_p - q'_1 - \cdots - q'_{p-1}|\langle q_1\rangle^{2\alpha}\cdots\langle q_p\rangle^{2\alpha}\langle q'_1\rangle^{2\alpha}\cdots\langle q'_{p-1}\rangle^{2\alpha}}, \tag{3.5.11}$$

where

$$C_\alpha := \int_{\mathbb{R}} \frac{d\zeta}{\langle\zeta\rangle^{2\alpha}}. \tag{3.5.12}$$

C_α is finite when $\alpha > \frac{1}{2}$ (Note $\alpha > 1 - \frac{1}{2(2p_0-1)} \ge 1 - \frac{1}{2(2p-1)} \ge \frac{1}{2}$). Following [7], in order to bound J_1, we introduce a non-negative spherically symmetric function h with rapid decay away from the unit ball in $\mathbb{R}^2$, such that $\check{h}(x) \ge 0$ decays fast outside the unit ball in $\mathbb{R}^2$, and

$$\frac{1}{\langle q\rangle^{2\alpha}} < \big(h * \frac{1}{|\cdot|^{2\alpha}}\big)(q). \tag{3.5.13}$$

Such a function h does exist. For example, we can take $h(y) = c_1 e^{-c_2 y^2}$ with appropriate c_1, c_2. Here we need $\alpha < 1$ for $h * \frac{1}{|\cdot|^{2\alpha}}$ to stay in $L^\infty(\mathbb{R}^2)$. Then ($d = 2$ below):

$$
\begin{aligned}
J_1 &< C_\alpha C \langle \big(\frac{1}{|\cdot|} * (h * \frac{1}{|\cdot|^{2\alpha}})\big) * (h * \frac{1}{|\cdot|^{2\alpha}}) \cdots * (h * \frac{1}{|\cdot|^{2\alpha}}), (h * \frac{1}{|\cdot|^{2\alpha}}) \rangle_{L^2(\mathbb{R}^d)} \\
&= C_\alpha C \int dx \big(\frac{1}{|\cdot|}\big)^\vee(x) \Big((h * \frac{1}{|\cdot|^{2\alpha}})^\vee(x) \Big)^{2p-1} \\
&= C_\alpha C' \int dx \frac{1}{|x|^{d-1}} \big(\check{h}(x)\big)^{2p-1} \big(\frac{1}{|x|^{d-2\alpha}}\big)^{2p-1} \\
&= C''_\alpha < \infty.
\end{aligned}
\tag{3.5.14}
$$

Thanks to the decay property of $\check{h}(x)$ outside of the unit ball, the only singularity of the above integral is the origin. Thus (3.5.14) holds if

$$
d - 1 + (2p-1)(d - 2\alpha) < d \quad \Longleftrightarrow \quad \alpha > \frac{d}{2} - \frac{1}{2(2p-1)} \tag{3.5.15}
$$

for all $1 \le p \le p_0$.

When $d = 2$, we need $\alpha > 1 - \frac{1}{2(p_0 - 1)}$ to yield (3.5.14). Terms $J_2, \cdots, J_{p+1}$ can be bounded in the same manner, thus it suffices to choose $C_\alpha = (p_0 + 1) C''_\alpha$. Theorem 3.5 is actually a substitution of Theorem 3.4 for high dimensions. $\square$

3.6 Uniqueness

We are ready to establish uniqueness theorems using results in previous sections. For that, we introduce some notations.

The infinite hierarchy (3.1.7) can be rewritten in integral form as

$$\gamma^{(k)}(t,\cdot) = U^{(k)}(t)\gamma^{(k)}(0,\cdot) - i\sum_{p=1}^{p_0} b_p \sum_{j=1}^{k} \int_0^t ds U^{(k)}(t-s) B_{j;k+1,\dots,k+p}\gamma^{(k+p)}(s,\cdot). \tag{3.6.1}$$

Here, $b_p = \int_{\mathbb{R}^{pd}} V^{(p)}(x)dx$. Also recall that the *free propagator* $U^{(k)}(t)$ is given by

$$U^{(k)}(t)\gamma^{(k)} = e^{it\Delta_{\pm}^{(k)}}\gamma^{(k)}$$

with $\Delta_{\pm}^{(k)} = \Delta_{\mathbf{x}_k} - \Delta_{\mathbf{x}'_k}$.

Now assume the initial condition $\gamma^{(k)}(0,\cdot) = 0$. For fixed positive integer k, thanks to Duhamel formula, we can write $\gamma^{(k)}$ in terms of the future iterates $\gamma^{(k+p_1)}$, $\gamma^{(k+p_1+p_2)}, \dots, \gamma^{(k+p_1+\dots+p_n)}$, where $p_1, p_2, \cdots p_n$ are integers chosen from set

$$S_{p_0} := \{1, 2, 3, \cdots, p_0\}.$$

Also let Q_j be half of the running sum over $p_1, p_2, p_3, \cdots$:

$$Q_j := p_1 + p_2 + \cdots + p_j \le p_0 j, \qquad j = 1, 2, \cdots$$

By convention let $Q_0 = 0$. Then we have

$$\begin{aligned}
\gamma^{(k)}(t_k,\cdot) &= \sum_{p_1\in S_{p_0}} b_{p_1}\int_0^{t_k} e^{i(t_k - t_{k+Q_1})\Delta_{\pm}^{(k)}} B_{k+Q_1}^{k}\big(\gamma^{(k+Q_1)}(t_{k+Q_1})\big)dt_{k+Q_1} \\
&= \sum_{p_1,p_2\in S_{p_0}} b_{p_1}b_{p_2}\int_0^{t_k} e^{i(t_k - t_{k+Q_1})\Delta_{\pm}^{(k)}} B_{k+Q_1}^{k} \\
&\qquad \Big(\int_0^{t_{k+Q_1}} e^{i(t_{k+Q_1} - t_{k+Q_2})\Delta_{\pm}^{(k+Q_1)}} B_{k+Q_2}^{k+Q_1}\big(\gamma^{(k+Q_2)}(t_{k+Q_2})\big)dt_{k+Q_1}\Big)dt_{k+Q_2} \\
&= \cdots
\end{aligned}$$

$$= \sum_{p_1,\cdots,p_n\in S_{p_0}} \Big(\prod_{j=1}^{n} b_{p_j}\Big) \int_0^{t_k} \cdots \int_0^{t_{k+Q_{n-1}}} J^k(\underline{t}_{k+Q_n}) dt_{k+Q_1} \ldots dt_{k+Q_n}, \tag{3.6.2}$$

where

$$\underline{t}_{k+Q_n} = (t_k, t_{k+Q_1}, \ldots, t_{k+Q_n}),$$

$$B^{k+Q_{j-1}}_{k+Q_j} := \sum_{j=1}^{k+Q_{n-1}} B_{j;k+Q_{n-1}+1,k+Q_{n-1}+2,\cdots,k+Q_n},$$

$$J^k(\underline{t}_{k+Q_n}) :=$$

$$e^{i(t_k-t_{k+Q_1})\Delta^{(k)}_{\pm}} B^k_{k+Q_1} \cdots e^{i(t_{k+Q_{n-1}}-t_{k+Q_n})\Delta^{(k+Q_{n-1})}_{\pm}} B^{k+Q_{n-1}}_{k+Q_n} \big(\gamma^{(k+Q_n)}(t_{k+Q_n})\big).$$

We present our main uniqueness theorems for $d = 1, 2$ in two separate subsections.

3.6.1 Uniqueness in 1D

For $d = 1$ we have the following theorem:

Theorem 3.6 (Uniqueness in 1D)**.** *Assume that $d = 1$, $t \in [0, T]$ and $\frac{1}{2} < \alpha \leq 1$. The maximal potential constant $b_0 = \max\{b_0^{(1)}, b_0^{(2)}, \cdots, b_0^{(p_0)}\}$ is positive and finite. Then we have*

$$\Big\| S^{(k,\alpha)} \gamma^{(k)}(t, \cdot) \Big\|_{L^2(\mathbb{R}^k\times\mathbb{R}^k)} \leq C^k (C_0 T)^n \tag{3.6.3}$$

for arbitrary n and constants C, C_0 that are depending on b_0, p_0, κ and α, but are independent of k and T.

Proof of Theorem 3.6. The idea of the proof is an iterative applications of spacial bound (3.4.18) and Cauchy-Schwarz inequality and at last followed by the use of the bound (3.4.6). Noticed that α is a constant in $(\frac{1}{2}, 1]$ and $e^{i(t_k - t_{k+Q_1})\Delta_{\pm}^{(k)}}$ is a unitary operator and commutes with the operator $S^{(k,\alpha)}$, thus we have

$$\begin{aligned}
&\left\| S^{(k,\alpha)} \int_0^{t_k} \cdots \int_0^{t_{k+Q_{n-1}}} J^k(\underline{t}_{k+Q_n}) dt_{k+Q_1} \dots dt_{k+Q_n} \right\|_{L^2(\mathbb{R}^{2k})} \\
&\leq \int_0^{t_k} \cdots \int_0^{t_{k+Q_{n-1}}} \Big\| S^{(k,\alpha)} e^{i(t_k - t_{k+Q_1})\Delta_{\pm}^{(k)}} B^k_{k+Q_1} e^{i(t_{k+Q_1} - t_{k+Q_2})\Delta_{\pm}^{(k+Q_1)}} B^{k+Q_1}_{k+Q_2} \cdots \\
&\quad \times e^{i(t_{k+Q_{n-1}} - t_{k+Q_n})\Delta_{\pm}^{(k+Q_{n-1})}} B^{k+Q_{n-1}}_{k+Q_n} \big(\gamma^{(k+Q_n)}(t_{k+Q_n}) \big) \Big\|_{L^2(\mathbb{R}^{2k})} dt_{k+Q_1} \dots dt_{k+Q_n} \\
&= \int_0^{t_k} \cdots \int_0^{t_{k+Q_{n-1}}} \Big\| S^{(k,\alpha)} B^k_{k+Q_1} e^{i(t_{k+Q_1} - t_{k+Q_2})\Delta_{\pm}^{(k+Q_1)}} B^{k+Q_1}_{k+Q_2} \cdots \\
&\quad \times e^{i(t_{k+Q_{n-1}} - t_{k+Q_n})\Delta_{\pm}^{(k+Q_{n-1})}} B^{k+Q_{n-1}}_{k+Q_n} \big(\gamma^{(k+Q_n)}(t_{k+Q_n}) \big) \Big\|_{L^2(\mathbb{R}^{2k})} dt_{k+Q_1} \dots dt_{k+Q_n} \\
&\leq k C_\alpha \int_0^{t_k} \cdots \int_0^{t_{k+Q_{n-1}}} \Big\| S^{(k+Q_1,\alpha)} e^{i(t_{k+Q_1} - t_{k+Q_2})\Delta_{\pm}^{(k+Q_1)}} B^{k+Q_1}_{k+Q_2} \cdots \\
&\quad \times e^{i(t_{k+Q_{n-1}} - t_{k+Q_n})\Delta_{\pm}^{(k+Q_{n-1})}} B^{k+Q_{n-1}}_{k+Q_n} \big(\gamma^{(k+Q_n)}(t_{k+Q_n}) \big) \Big\|_{L^2(\mathbb{R}^{2(k+Q_1)})} dt_{k+Q_1} \dots dt_{k+Q_n}
\end{aligned} \tag{3.6.4}$$

$$\begin{aligned}
&\leq \cdots \\
&\leq \prod_{j=0}^{n-2} \Big((k+Q_j) C_\alpha \Big) \int_0^{t_k} \cdots \int_0^{t_{k+Q_{n-1}}} \Big\| S^{(k+Q_{n-1},\alpha)} e^{i(t_{k+Q_{n-1}} - t_{k+Q_n})\Delta_{\pm}^{(k+Q_{n-1})}}
\end{aligned} \tag{3.6.5}$$

$$\begin{aligned}
&\quad \times B^{k+Q_{n-1}}_{k+Q_n} \big(\gamma^{(k+Q_n)}(t_{k+Q_n}) \big) \Big\|_{L^2\left(\mathbb{R}^{2(k+Q_{n-1})}\right)} dt_{k+Q_1} \dots dt_{k+Q_n} \\
&= \prod_{j=0}^{n-2} \Big((k+Q_j) C_\alpha \Big) \int_0^{t_k} \cdots \int_0^{t_{k+Q_{n-1}}} \Big\| S^{(k+Q_{n-1},\alpha)} \\
&\quad \times B^{k+Q_{n-1}}_{k+Q_n} \big(\gamma^{(k+Q_n)}(t_{k+Q_n}) \big) \Big\|_{L^2\left(\mathbb{R}^{2(k+Q_{n-1})}\right)} dt_{k+Q_1} \dots dt_{k+Q_n}
\end{aligned}$$

$$\leq C_\alpha^{n-1} \prod_{j=0}^{n-1} (k + p_0 j) \int_0^{t_k} \cdots \int_0^{t_{k+Q_{n-1}}} C^{k+p_0 n} dt_{k+Q_1} \ldots dt_{k+Q_n} \tag{3.6.6}$$

$$\leq p_0^n C_\alpha^{n-1} \lceil \frac{k}{p_0} \rceil (\lceil \frac{k}{p_0} \rceil + 1) \cdots (\lceil \frac{k}{p_0} \rceil + n - 1) C^{k+p_0 n} \frac{t_k^n}{n!}$$

$$= C^{k+p_0 n} p_0^n C_\alpha^{n-1} \binom{\lceil \frac{k}{p_0} \rceil + n - 1}{n} t_k^n$$

$$\leq C^k (p_0 C^{p_0} t_k)^n C_\alpha^{n-1} 2^{\lceil \frac{k}{p_0} \rceil + n - 1}.$$

Thus

$$\left\| S^{(k,\alpha)} \gamma^{(k)}(t_k, \cdot) \right\|_{L^2(\mathbb{R}^k \times \mathbb{R}^k)}$$

$$\leq \sum_{p_1, \cdots, p_n \in S_{p_0}} \left(\prod_{j=1}^n b_0^{(p_j)} \right) \left\| S^{(k,\alpha)} \int_0^{t_k} \cdots \int_0^{t_{k+Q_{n-1}}} J^k(\underline{t}_{k+Q_n}) dt_{k+Q_1} \ldots dt_{k+Q_n} \right\|_{L^2(\mathbb{R}^{2k})}$$

$$\leq p_0 (b_0)^n C^k (p_0 C^{p_0} t_k)^n C_\alpha^{n-1} 2^{\lceil \frac{k}{p_0} \rceil + n - 1}$$

$$\leq C^k (C_0 T)^n,$$

where (3.6.4) is based on (3.4.18) and we keep using (3.4.18) to obtain (3.6.5). Since $e^{i(t_{k+Q_{n-1}} - t_{k+Q_n}) \Delta_\pm^{(k+Q_{n-1})}}$ is unitary and commutes with $S^{(k+Q_{n-1},\alpha)}$, then after applying Theorem 3.3, we have (3.6.6). Here, $\lceil x \rceil$ is the ceiling function. In the last line, we choose appropriate C and C_0 to finish the proof. $\square$

3.6.2 Uniqueness in 2D

For $d = 2$ we have the following theorem

Theorem 3.7 (Uniqueness in 2D). *Assume that $d = 2$ and $t \in [0, T]$, $1 - \frac{1}{2(2p_0 - 1)} < \alpha \leq 1$. The maximal potential constant $b_0 = \max\{b_0^{(1)}, b_0^{(2)}, \cdots, b_0^{(p_0)}\}$*

is positive and finite. Then we have

$$\left\| S^{(k,\alpha)}\gamma^{(k)}(t,\cdot) \right\|_{L^2(\mathbb{R}^{2k}\times\mathbb{R}^{2k})} \leq C^k (C_0\sqrt{T})^n \tag{3.6.7}$$

for arbitrary n *and constants* C, C_0 *that are depending on* b_0*,* p_0*,* κ *and* α*, but are independent of* k *and* T*.*

Based on the above theorems 3.6 and 3.7, if we are given sufficiently small T, then for all $t \in [0,T]$:

$$\left\| S^{(k,\alpha)}\gamma^{(k)}(t,\cdot) \right\|_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} \to 0 \text{ as } n \to \infty. \tag{3.6.8}$$

This implies that $\gamma^{(k)}(t,\cdot) = 0$. Since k is arbitrary, therefore solutions to the infinite hierarchy (3.1.7) with zero initial conditions are unique in the above norm.

The rest of this chapter is devoted to the proof of Theorem 3.7.

3.7 Combinatorial argument

One of the main ingredients in the proof of Theorem 3.7 is an Erdös-Schlein-Yau type combinatorial argument which is inspired by Klainerman-Machedon [34]. We will develop all the details of the reduction procedure in this section.

3.7.1 Graphical representations

The key point in the proof of Theorem 3.7 is to handle the iterative terms from Duhamel formula. Throughout this section, we will prove several auxiliary lemmas to help us group these terms and also derive some bounds on certain equivalence classes. The technique we use here is inspired by [34, 7], and adapted to a much more general setting.

For the reader's convenience, we recall some notation defined before:

$$\forall 1 \le j \le n, \quad p_j \in S_{p_0} = \{1, 2, 3, \cdots, p_0\}.$$

$$Q_j := p_1 + p_2 + \cdots + p_j, \qquad j = 1, 2, \cdots$$

Also, $B_{k+Q_n}^{k+Q_{n-1}} = \sum_{j=1}^{k+Q_{n-1}} B_{j;k+Q_{n-1}+1,\cdots,k+Q_n}$, we can rewrite $J^k(\underline{t}_{k+Q_n})$ as the following:

$$J^k(\underline{t}_{k+Q_n}) = \sum_{\mu \in M} J^k(\underline{t}_{k+Q_n}; \mu) \tag{3.7.1}$$

where

$$\begin{aligned} J^k(\underline{t}_{k+Q_n}; \mu) = & e^{i(t_k - t_{k+Q_1})\Delta_\pm^{(k)}} B_{\mu(k+1);k+1,\cdots,k+Q_1} e^{i(t_{k+Q_1} - t_{k+Q_2})\Delta_\pm^{(k+Q_1)}} \cdots \\ & \times e^{i(t_{k+Q_{n-1}} - t_{k+Q_n})\Delta_\pm^{(k+Q_{n-1})}} B_{\mu(k+Q_{n-1}+1);k+Q_{n-1}+1,\cdots,k+Q_n} \left(\gamma^{(k+Q_n)}(t_{k+Q_n})\right) \end{aligned} \tag{3.7.2}$$

and μ is a map from $\{k+1, k+2, \ldots, k+Q_{n-1}+1\}$ to $\{1, 2, \ldots, k+Q_{n-1}\}$ such that $\mu(2) = 1$ and $\mu(j) < j$ for all j. M is the set of all these mappings.

By the definition of μ, we can represent it by highlighting exactly one nonzero entry in each column of a $(k+Q_{n-1}) \times n$ matrix of the form:

$$\begin{pmatrix} \mathbf{B_{1;k+1,\cdots,k+Q_1}} & B_{1;k+Q_1+1,\cdots,k+Q_2} & \cdots & \mathbf{B_{1;k+Q_{n-1}+1,\cdots,k+Q_n}} \\ B_{2;k+1,\cdots,k+Q_1} & \mathbf{B_{2;k+Q_1+1,\cdots,k+Q_2}} & \cdots & B_{2;k+Q_{n-1}+1,\cdots,k+Q_n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{k;k+1,\cdots,k+Q_1} & B_{k;k+Q_1+1,\cdots,k+Q_2} & \cdots & B_{k;k+Q_{n-1}+1,\cdots,k+Q_n} \\ 0 & B_{k+1;k+Q_1+1,\cdots,k+Q_2} & \cdots & B_{k+1;k+Q_{n-1}+1,\cdots,k+Q_n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{k+Q_{n-1};k+Q_{n-1}+1,\cdots,k+Q_n} \end{pmatrix}. \tag{3.7.3}$$

Henceforth, we can rewrite (3.6.2) as

$$\gamma^{(k)}(t_k,\cdot) = \int_0^{t_k} \cdots \int_0^{t_{k+Q_{n-1}}} \sum_{\mu\in M} J^k(\underline{t}_{k+Q_n};\mu)dt_{k+Q_1}\dots dt_{k+Q_n}. \tag{3.7.4}$$

So the basic term of the above sum is the following integral

$$I(\mu,\sigma) = \int_{t_k\geq t_{\sigma(k+Q_1)}\geq\cdots\geq t_{\sigma(k+Q_n)}} J^k(\underline{t}_{k+Q_n};\mu)dt_{k+Q_1}\dots dt_{k+Q_n}, \tag{3.7.5}$$

where σ is a permutation of $k+Q_1, k+Q_2, \dots, k+Q_n$. We will associate the integral $I(\mu,\sigma)$ to the following $(k+Q_{n-1}+1)\times n$ matrix. Matrix (3.7.6) is also helpful to visualize $B_{\mu(k+Q_{j-1}+1);k+Q_{j-1}+1,\cdots,k+Q_j}$, $j=1,2,\dots,n$ and σ:

$$\begin{pmatrix}
t_{\sigma^{-1}(k+Q_1)} & t_{\sigma^{-1}(k+Q_2)} & \cdots & t_{\sigma^{-1}(k+Q_n)} \\
\mathbf{B_{1;k+1,\cdots,k+Q_1}} & B_{1;k+Q_1+1,\cdots,k+Q_2} & \cdots & \mathbf{B_{1;k+Q_{n-1}+1,\cdots,k+Q_n}} \\
B_{2;k+1,\cdots,k+Q_1} & \mathbf{B_{2;k+Q_1+1,\cdots,k+Q_2}} & \cdots & B_{2;k+Q_{n-1}+1,\cdots,k+Q_n} \\
\cdots & \cdots & \cdots & \cdots \\
B_{k;k+1,\cdots,k+Q_1} & B_{k;k+Q_1+1,\cdots,k+Q_2} & \cdots & B_{k;k+Q_{n-1}+1,\cdots,k+Q_n} \\
0 & B_{k+1;k+Q_1+1,\cdots,k+Q_2} & \cdots & B_{k+1;k+Q_{n-1}+1,\cdots,k+Q_n} \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & B_{k+Q_{n-1};k+Q_{n-1}+1,\cdots,k+Q_n}
\end{pmatrix}. \tag{3.7.6}$$

We label the columns of matrix (3.7.6) by 1 through n while rows 0 through $k+Q_{n-1}$.

3.7.2 Acceptable moves

It is an important step to introduce the so-called "acceptable move" on the set of matrices of the form (3.7.6). In particular, if $\mu(k+Q_j+1) < \mu(k+Q_{j-1}+1)$, we can perform the following changes at the same time:

- exchange the highlights in columns j and $j+1$;

- exchange the highlights in rows $k+Q_{j-1}+1$ and $k+Q_j+1$;
- exchange the highlights in rows $k+Q_{j-1}+2$ and $k+Q_j+2$;
- $\cdots$
- exchange the highlights in rows $k+Q_{j-1}+r_0$ and $k+Q_j+r_0$;
- exchange $t_{\sigma^{-1}(k+Q_j)}$ and $t_{\sigma^{-1}(k+Q_{j+1})}$;

with $r_0 := \min\{p_j, p_{j+1}\}$.

For instance, if $k=1, n=4, p_1=2, p_2=1, p_3=3, p_4=2$, then we go from

$$\begin{pmatrix} t_{\sigma^{-1}(1+Q_1)} & t_{\sigma^{-1}(1+Q_2)} & t_{\sigma^{-1}(1+Q_3)} & t_{\sigma^{-1}(1+Q_4)} \\ \mathbf{B_{1;2,3}} & B_{1;4} & B_{1;5,6,7} & B_{1;8,9} \\ 0 & B_{2;4} & \mathbf{B_{2;5,6,7}} & B_{2;8,9} \\ 0 & \mathbf{B_{3;4}} & B_{3;5,6,7} & B_{3;8,9} \\ 0 & 0 & B_{4;5,6,7} & \mathbf{B_{4;8,9}} \\ 0 & 0 & 0 & B_{5;8,9} \\ 0 & 0 & 0 & B_{6;8,9} \\ 0 & 0 & 0 & B_{7;8,9} \end{pmatrix}$$

to

$$\begin{pmatrix} t_{\sigma^{-1}(1+Q_1)} & t_{\sigma^{-1}(1+Q_3)} & t_{\sigma^{-1}(1+Q_2)} & t_{\sigma^{-1}(1+Q_4)} \\ \mathbf{B_{1;2,3}} & B_{1;4} & B_{1;5,6,7} & B_{1;8,9} \\ 0 & \mathbf{B_{2;4}} & B_{2;5,6,7} & B_{2;8,9} \\ 0 & B_{3;4} & \mathbf{B_{3;5,6,7}} & B_{3;8,9} \\ 0 & 0 & B_{4;5,6,7} & B_{4;8,9} \\ 0 & 0 & 0 & \mathbf{B_{5;8,9}} \\ 0 & 0 & 0 & B_{6;8,9} \\ 0 & 0 & 0 & B_{7;8,9} \end{pmatrix}.$$

The reason for taking such moves is explained by the following lemma.

Lemma 3.8. *Let (μ, σ) be transformed into (μ', σ') by an acceptable move. Then, for the corresponding integrals* (3.7.5), *$I(\mu, \sigma) = I(\mu', \sigma')$.*

Proof. The proof, as in [34] and [7], is straightforward but somewhat tedious. We modify the proof of Lemma 7.1 in [7] so that it can be used here. Since there is only one acceptable move between the two integrals, most part of their expressions share the same terms. Let us fix $j \geq 3$, select two integers i, l such that $i < l < j < j+1$ and compare $I(\mu, \sigma)$ and $I(\mu', \sigma')$

$$\begin{aligned} I(\mu,\sigma) &= \int_{t_k \geq \cdots t_{\sigma(k+Q_j)} \geq t_{\sigma(k+Q_{j+1})} \cdots \geq t_{\sigma(k+Q_n)} \geq 0} J^k(\underline{t}_{k+Q_n}; \mu) dt_{k+Q_1} \ldots dt_{k+Q_n} \\ &= \int_{t_k \geq \cdots t_{\sigma(k+Q_j)} \geq t_{\sigma(k+Q_{j+1})} \cdots \geq t_{\sigma(k+Q_n)} \geq 0} \cdots e^{i(t_{k+Q_{j-1}} - t_{k+Q_j})\Delta_{\pm}^{(k+Q_{j-1})}} \\ &\quad \times B_{l;k+Q_{j-1}+1,\cdots,k+Q_j} e^{i(t_{k+Q_j} - t_{k+Q_{j+1}})\Delta_{\pm}^{(k+Q_j)}} B_{i;k+Q_j+1,\cdots,k+Q_{j+1}} \\ &\quad \times e^{i(t_{k+Q_{j+1}} - t_{k+Q_{j+2}})\Delta_{\pm}^{(k+Q_{j+1})}} (\cdots) dt_{k+Q_1} \ldots dt_{k+Q_n} \end{aligned} \tag{3.7.7}$$

and

$$\begin{aligned} I(\mu',\sigma') &= \int_{t_k \geq \cdots t_{\sigma'(k+Q_j)} \geq t_{\sigma'(k+Q_{j+1})} \cdots \geq t_{\sigma'(k+Q_n)} \geq 0} J^k(\underline{t}_{k+Q_n}; \mu') dt_{k+Q_1} \ldots dt_{k+Q_n} \\ &= \int_{t_k \geq \cdots t_{\sigma'(k+Q_j)} \geq t_{\sigma'(k+Q_{j+1})} \cdots \geq t_{\sigma'(k+Q_n)} \geq 0} \cdots e^{i(t_{k+Q_{j-1}} - t_{k+Q_j})\Delta_{\pm}^{(k+Q_{j-1})}} \\ &\quad \times B_{i;k+Q_{j-1}+1,\cdots,k+Q_j} e^{i(t_{k+Q_j} - t_{k+Q_{j+1}})\Delta_{\pm}^{(k+Q_j)}} B_{l;k+Q_j+1,\cdots,k+Q_{j+1}} \\ &\quad \times e^{i(t_{k+Q_{j+1}} - t_{k+Q_{j+2}})\Delta_{\pm}^{(k+Q_{j+1})}} (\cdots)' dt_{k+Q_1} \ldots dt_{k+Q_n}. \end{aligned} \tag{3.7.8}$$

The omitted expression "$\cdots$" in (3.7.7) and (3.7.8) coincide.

For $1 \leq r \leq r_0 = \min\{p_j, p_{j+1}\}$, $s \geq Q_j$ and index m: $j+1 \leq m \leq n$, any

$B_{k+Q_{j-1}+r;s+1,\cdots,s+p_m}$ (when it is highlighted) in $(\cdots)$ of (3.7.7) will become $B_{k+Q_j+r;s+1,\cdots,s+p_m}$ in $(\cdots)'$ of (3.7.8) and any $B_{k+Q_j+r;s+1,\cdots,s+p_m}$ (when it is highlighted) in $(\cdots)$ of (3.7.8) become $B_{k+Q_{j-1}+r;s+1,\cdots,s+p_m}$ in $(\cdots)'$ of (3.7.7);

All the changes are illustrated in the table below:

$(\cdots)$	$\longleftrightarrow$	$(\cdots)'$
$B_{k+Q_{j-1}+1;s+1,\cdots,s+p_m}$	$\leftrightarrow$	$B_{k+Q_j+1;s+1,\cdots,s+p_m}$
$B_{k+Q_{j-1}+2;s+1,\cdots,s+p_m}$	$\leftrightarrow$	$B_{k+Q_j+2;s+1,\cdots,s+p_m}$
$\vdots$	$\leftrightarrow$	$\vdots$
$B_{k+Q_{j-1}+r_0;s+1,\cdots,s+p_m}$	$\leftrightarrow$	$B_{k+Q_j+r_0;s+1,\cdots,s+p_m}$

Recall that our goal is to prove $I(\mu,\sigma) = I(\mu',\sigma')$. Let P and $\tilde{P}$ be as:

$$P = B_{l;k+Q_{j-1}+1,\cdots,k+Q_j} e^{i(t_{k+Q_j}-t_{k+Q_{j+1}})\Delta_\pm^{(k+Q_j)}} B_{i;k+Q_j+1,\cdots,k+Q_{j+1}}, \tag{3.7.9}$$

$$\tilde{P} = B_{i;k+Q_j+1,\cdots,k+Q_{j+1}} e^{-i(t_{k+Q_j}-t_{k+Q_{j+1}})\tilde{\Delta}_\pm^{(k+Q_j)}} B_{l;k+Q_{j-1}+1,\cdots,k+Q_j}, \tag{3.7.10}$$

where

$$\tilde{\Delta}_\pm^{(k+Q_j)} = \Delta_\pm^{(k+Q_j)} - \Delta_{\pm,x_{k+Q_j}} - \Delta_{\pm,x_{k+Q_j-1}} - \cdots - \Delta_{\pm,x_{k+Q_{j-1}+1}}$$
$$+ \Delta_{\pm,x_{k+Q_j+1}} + \Delta_{\pm,x_{k+Q_j+2}} + \cdots + \Delta_{\pm,x_{k+Q_{j+1}}}.$$

We've used this notion above: $\Delta_{\pm,x_j} = \Delta_{x_j} - \Delta_{x'_j}$.

We will show that

$$\begin{aligned} &e^{i(t_{k+Q_{j-1}}-t_{k+Q_j})\Delta_\pm^{(k+Q_{j-1})}} P e^{i(t_{k+Q_{j+1}}-t_{k+Q_{j+2}})\Delta_\pm^{(k+Q_{j+1})}} \\ &= e^{i(t_{k+Q_{j-1}}-t_{k+Q_{j+1}})\Delta_\pm^{(k+Q_{j-1})}} \tilde{P} e^{i(t_{k+Q_j}-t_{k+Q_{j+2}})\Delta_\pm^{(k+Q_{j+1})}}. \end{aligned} \tag{3.7.11}$$

Indeed in (3.7.9) we can write $\Delta_\pm^{(k+Q_j)} = \Delta_{\pm,x_i} + (\Delta_\pm^{(k+Q_j)} - \Delta_{\pm,x_i})$. Therefore,

$$e^{i(t_{k+Q_j}-t_{k+Q_{j+1}})\Delta_\pm^{(k+Q_j)}} = e^{i(t_{k+Q_j}-t_{k+Q_{j+1}})\Delta_{\pm,x_i}} e^{i(t_{k+Q_j}-t_{k+Q_{j+1}})(\Delta_\pm^{(k+Q_j)}-\Delta_{\pm,x_i})}.$$

Observe that the first term on the RHS of the above equation can be commuted to the left of $B_{l;k+Q_{j-1}+1,\cdots,k+Q_j}$ and the second term to the right of $B_{i;k+Q_j+1,\cdots,k+Q_{j+1}}$, thus, after two commutations,

$$\begin{aligned}P =& e^{i(t_{k+Q_j}-t_{k+Q_{j+1}})\Delta_{\pm,x_i}} B_{l;k+Q_{j-1}+1,\cdots,k+Q_j} B_{i;k+Q_j+1,\cdots,k+Q_{j+1}} \\ &\times e^{i(t_{k+Q_j}-t_{k+Q_{j+1}})(\Delta_\pm^{(k+Q_j)}-\Delta_{\pm,x_i})}\end{aligned} \tag{3.7.12}$$

and the LHS of (3.7.11) becomes

$$\begin{aligned}&e^{i(t_{k+Q_{j-1}}-t_{k+Q_j})\Delta_\pm^{(k+Q_{j-1})}} P e^{i(t_{k+Q_{j+2}}-t_{k+Q_{j+2}})\Delta_\pm^{(k+Q_{j+1})}} \\ &= e^{i(t_{k+Q_{j-1}}-t_{k+Q_j})\Delta_\pm^{(k+Q_{j-1})}} e^{i(t_{k+Q_j}-t_{k+Q_{j+1}})\Delta_{\pm,x_i}} B_{l;k+Q_{j-1}+1,\cdots,k+Q_j} \\ &\quad\times B_{i;k+Q_j+1,\cdots,k+Q_{j+1}} e^{i(t_{k+Q_j}-t_{k+Q_{j+1}})(\Delta_\pm^{(k+Q_j)}-\Delta_{\pm,x_i})} e^{i(t_{k+Q_{j+1}}-t_{k+Q_{j+2}})\Delta_\pm^{(k+Q_{j+1})}} \\ &= e^{i(t_{k+Q_{j-1}}-t_{k+Q_j})\Delta_\pm^{(k+Q_{j-1})}} e^{i(t_{k+Q_j}-t_{k+Q_{j+1}})\Delta_{\pm,x_i}} B_{i;k+Q_j+1,\cdots,k+Q_{j+1}} \\ &\quad\times B_{l;k+Q_{j-1},\cdots,k+Q_j} e^{i(t_{k+Q_{j+1}}-t_{k+Q_{j+2}})(\Delta_{\pm,x_i}+\Delta_{\pm,x_{k+Q_j+1}}+\cdots+\Delta_{\pm,x_{k+Q_{j+1}}})} \\ &\quad\times e^{i(t_{k+Q_j}-t_{k+Q_{j+2}})(\Delta_{\pm,x_1}+\cdots+\hat{\Delta}_{\pm,x_i}+\cdots+\Delta_{\pm,x_{k+Q_j}})},\end{aligned} \tag{3.7.13}$$

where a hat denotes a missing term.

Similarly, we can rewrite $\tilde{\Delta}_\pm^{(k+Q_j)}$ as

$$\begin{aligned}\tilde{\Delta}_\pm^{(k+Q_j)} &= \Delta_\pm^{(k+Q_j)} - \Delta_{\pm,x_{k+Q_j}} - \cdots - \Delta_{\pm,x_{k+Q_{j-1}+1}} \\ &\qquad + \Delta_{\pm,x_{k+Q_j+1}} + \cdots + \Delta_{\pm,x_{k+Q_{j+1}}} \\ &= \Delta_\pm^{(k+Q_{j-1})} + \Delta_{\pm,x_{k+Q_j+1}} + \cdots + \Delta_{\pm,x_{k+Q_{j+1}}} \\ &= (\Delta_\pm^{(k+Q_{j-1})} - \Delta_{\pm,x_i}) + (\Delta_{\pm,x_i} + \Delta_{\pm,x_{k+Q_j+1}} + \cdots + \Delta_{\pm,x_{k+Q_{j+1}}}).\end{aligned}$$

Hence the factor $e^{-i(t_{k+Q_j}-t_{k+Q_{j+1}})\tilde{\Delta}_\pm^{(k+Q_j)}}$ appearing in the definition of

$\tilde{P}$ can be rewritten as

$$
\begin{aligned}
& e^{-i(t_{k+Q_j}-t_{k+Q_{j+1}})\tilde{\Delta}_{\pm}^{(k+Q_j)}} \\
& = e^{-i(t_{k+Q_j}-t_{k+Q_{j+1}})(\Delta_{\pm}^{(k+Q_{j-1})}-\Delta_{\pm,x_i})} \\
& \times e^{-i(t_{k+Q_j}-t_{k+Q_{j+1}})(\Delta_{\pm,x_i}+\Delta_{\pm,x_{k+Q_j+1}}+\cdots+\Delta_{\pm,x_{k+Q_{j+1}}})}.
\end{aligned}
$$

and consequently,

$$
\begin{aligned}
\tilde{P} = & e^{-i(t_{k+Q_j}-t_{k+Q_{j+1}})(\Delta_{\pm}^{(k+Q_{j-1})}-\Delta_{\pm,x_i})} B_{i;k+Q_j+1,\cdots,k+Q_{j+1}} B_{l;k+Q_{j-1}+1,\cdots,k+Q_j} \\
& \times e^{-i(t_{k+Q_j}-t_{k+Q_{j+1}})(\Delta_{\pm,x_i}+\Delta_{\pm,x_{k+Q_j+1}}+\cdots+\Delta_{\pm,x_{k+Q_{j+1}}})}.
\end{aligned}
\tag{3.7.14}
$$

The RHS of (3.7.11) equals to

$$
\begin{aligned}
& e^{i(t_{k+Q_{j-1}}-t_{k+Q_{j+1}})\Delta_{\pm}^{(k+Q_{j-1})}} \tilde{P} e^{i(t_{k+Q_j}-t_{k+Q_{j+2}})\Delta_{\pm}^{(k+Q_{j+1})}} \\
& = e^{i(t_{k+Q_{j-1}}-t_{k+Q_{j+1}})\Delta_{\pm}^{(k+Q_{j-1})}} e^{-i(t_{k+Q_j}-t_{k+Q_{j+1}})(\Delta_{\pm}^{(k+Q_{j-1})}-\Delta_{\pm,x_i})} \\
& \quad \times B_{i;k+Q_j+1,\cdots,k+Q_{j+1}} B_{l;k+Q_{j-1}+1,\cdots,k+Q_j} \\
& \quad \times e^{-i(t_{k+Q_j}-t_{k+Q_{j+1}})(\Delta_{\pm,x_i}+\Delta_{\pm,x_{k+Q_j+1}}+\cdots+\Delta_{\pm,x_{k+Q_{j+1}}})} e^{i(t_{k+Q_j}-t_{k+Q_{j+2}})\Delta_{\pm}^{(k+Q_{j+1})}} \\
& = e^{i(t_{k+Q_{j-1}}-t_{k+Q_j})\Delta_{\pm}^{(k+Q_{j-1})}} e^{i(t_{k+Q_j}-t_{k+Q_{j+1}})\Delta_{\pm,x_i}} B_{i;k+Q_j+1,\cdots,k+Q_{j+1}} \\
& \quad \times B_{l;k+Q_{j-1}+1,\cdots,k+Q_j} e^{i(t_{k+Q_{j+1}}-t_{k+Q_{j+2}})(\Delta_{\pm,x_i}+\Delta_{\pm,x_{k+Q_j+1}}+\cdots+\Delta_{\pm,x_{k+Q_{j+1}}})} \\
& \quad \times e^{i(t_{k+Q_j}-t_{k+Q_{j+2}})(\Delta_{\pm,x_1}+\cdots+\hat{\Delta}_{\pm,x_i}+\cdots+\Delta_{\pm,x_{k+Q_j}})},
\end{aligned}
\tag{3.7.15}
$$

which is the same as (3.7.13). So (3.7.11) is proved.

Note that $r_0 = \min\{p_j, p_{j+1}\}$. By the symmetry of $\gamma^{(k+Q_n)}$, we can perform the following exchanges without changing its value

- exchange $(x_{k+Q_{j-1}+1}, x'_{k+Q_{j-1}+1})$ with $(x_{k+Q_j+1}, x'_{k+Q_j+1})$;

- exchange $(x_{k+Q_{j-1}+2}, x'_{k+Q_{j-1}+2})$ with $(x_{k+Q_j+2}, x'_{k+Q_j+2})$;
- $\cdots$
- exchange $(x_{k+Q_{j-1}+r_0}, x'_{k+Q_{j-1}+r_0})$ with $(x_{k+Q_j+r_0}, x'_{k+Q_j+r_0})$;

After performing these exchanges only in the arguments of $\gamma^{(k+Q_n)}$ we can rewrite (3.7.7) based on (3.7.11) as follows:

$$
\begin{aligned}
&I(\mu,\sigma)\\
&= \int_{t_k\geq \cdots t_{\sigma(k+Q_j)}\geq t_{\sigma(k+Q_{j+1})}\cdots \geq t_{\sigma(k+Q_n)}\geq 0} \cdots\\
&\times e^{i(t_{k+Q_{j-1}}-t_{k+Q_j})\Delta_\pm^{(k+Q_{j-1})}} P e^{i(t_{k+Q_{j+1}}-t_{k+Q_{j+2}})\Delta_\pm^{(k+Q_{j+1})}} (\cdots)' dt_{k+Q_1}\dots dt_{k+Q_n}\\
&= \int_{t_k\geq \cdots t_{\sigma(k+Q_j)}\geq t_{\sigma(k+Q_{j+1})}\cdots \geq t_{\sigma(k+Q_n)}\geq 0} \cdots\\
&\times e^{i(t_{k+Q_{j-1}}-t_{k+Q_{j+1}})\Delta_\pm^{(k+Q_{j-1})}} \tilde{P} e^{i(t_{k+Q_j}-t_{k+Q_{j+2}})\Delta_\pm^{(k+Q_{j+1})}} (\cdots)' dt_{k+Q_1}\dots dt_{k+Q_n}\\
&= \int_{t_k\geq \cdots t_{\sigma(k+Q_j)}\geq t_{\sigma(k+Q_{j+1})}\cdots \geq t_{\sigma(k+Q_n)}\geq 0} \int_{\mathbb{R}^{(k+Q_{j+1})d}} \cdots e^{i(t_{k+Q_{j-1}}-t_{k+Q_{j+1}})\Delta_\pm^{(k+Q_{j-1})}}\\
&\quad \times \delta_{i;k+Q_j+1,\cdots,k+Q_{j+1}} e^{-i(t_{k+Q_j}-t_{k+Q_{j+1}})\tilde{\Delta}_\pm^{(k+Q_j)}} \delta_{l;k+Q_{j-1}+1,\cdots,k+Q_j}\\
&\quad \times e^{i(t_{k+Q_j}-t_{k+Q_{j+2}})\Delta_\pm^{(k+Q_{j+1})}} (\cdots)' dt_{k+Q_1}\dots dt_{k+Q_n},
\end{aligned}
\tag{3.7.16}
$$

in which $\delta_{j;s+1,\cdots,s+p_m}$ denotes the abbreviated kernel of the operator $B_{j;s+1,\cdots,s+p_m}$:

$$
\begin{aligned}
\delta_{j;s+1,\cdots,s+p_m} =& \delta(x_j - x_{s+1})\delta(x_j - x'_{s+1})\cdots\delta(x_j - x_{s+p_m})\delta(x_j - x'_{s+p_m})\\
&- \delta(x'_j - x_{s+1})\delta(x'_j - x'_{s+1})\cdots\delta(x'_j - x_{s+p_m})\delta(x'_j - x'_{s+p_m}).
\end{aligned}
\tag{3.7.17}
$$

In (3.7.16) we perform the following change of variables:

- exchange $(t_{k+Q_{j-1}+1}, x_{k+Q_{j-1}+1}, x'_{k+Q_{j-1}+1})$ with $(t_{k+Q_j+1}, x_{k+Q_j+1}, x'_{k+Q_j+1})$;
- exchange $(t_{k+Q_{j-1}+2}, x_{k+Q_{j-1}+2}, x'_{k+Q_{j-1}+2})$ with $(t_{k+Q_j+2}, x_{k+Q_j+2}, x'_{k+Q_j+2})$;
- $\cdots$
- exchange $(t_{k+Q_{j-1}+r_0}, x_{k+Q_{j-1}+r_0}, x'_{k+Q_{j-1}+r_0})$ with $(t_{k+Q_j+r_0}, x_{k+Q_j+r_0}, x'_{k+Q_j+r_0})$;

in the whole integral. Under the same change of variables $\tilde{\Delta}_{\pm}^{(k+Q_j)}$ becomes

$$\begin{aligned}\tilde{\Delta}_{\pm}^{(k+Q_j)} &= \Delta_{\pm}^{(k+Q_j)} - \Delta_{\pm,x_{k+Q_j}} - \cdots - \Delta_{\pm,x_{k+Q_{j-1}+1}} \\ &\qquad + \Delta_{\pm,x_{k+Q_j+1}} + \cdots + \Delta_{\pm,x_{k+Q_{j+1}}} \\ &= \Delta_{\pm}^{(k+Q_{j-1})} + \Delta_{\pm,x_{k+Q_j+1}} + \cdots + \Delta_{\pm,x_{k+Q_{j+1}}} \\ &\to \Delta_{\pm}^{(k+Q_{j-1})} + \Delta_{\pm,x_{k+Q_{j-1}+1}} + \cdots + \Delta_{\pm,x_{k+Q_j}} \\ &= \Delta_{\pm}^{(k+Q_j)}.\end{aligned}$$

Note that $\Delta_{\pm}^{(k+Q_{j+1})}$ remain unchanged under this change of variable. Therefore, we obtain:

$$\begin{aligned}I(\mu,\sigma) &= \int_{t_k \geq \cdots t_{\sigma'(k+Q_j)} \geq t_{\sigma'(k+Q_{j+1})} \cdots \geq t_{\sigma'(k+Q_n)}} \cdots e^{i(t_{k+Q_{j-1}} - t_{k+Q_j})\Delta_{\pm}^{(k+Q_{j-1})}} \\ &\qquad \times B_{i;k+Q_{j-1}+1,\cdots,k+Q_j} e^{i(t_{k+Q_j} - t_{k+Q_{j+1}})\Delta_{\pm}^{(k+Q_j)}} B_{l;k+Q_j+1,\cdots,k+Q_{j+1}} \\ &\qquad \times e^{i(t_{k+Q_{j+1}} - t_{k+Q_{j+2}})\Delta_{\pm}^{(k+Q_{j+1})}} (\cdots)' dt_{k+Q_1} \ldots dt_{k+Q_n} \\ &= I(\mu',\sigma'),\end{aligned} \tag{3.7.18}$$

where $\sigma' = (k+Q_j, k+Q_{j+1}) \circ \sigma$. $(k+Q_j, k+Q_{j+1})$ denotes the permutation which reverses $k+Q_j$ and $k+Q_{j+1}$. $\square$

Next, let us consider the subset $\{\mu_s\} \subset M$ of *special upper echelon* matrices in which each highlighted element of a higher row is to the left of each highlighted element of a lower row. A simple example of a special upper echelon matrix is given below (with $k=1, n=4, p_1=2, p_2=1, p_3=3, p_4=2$)

$$\begin{pmatrix} \mathbf{B_{1;2,3}} & \mathbf{B_{1;4}} & B_{1;5,6,7} & B_{1;8,9} \\ 0 & B_{2;4} & B_{2;5,6,7} & B_{2;8,9} \\ 0 & B_{3;4} & B_{3;5,6,7} & B_{3;8,9} \\ 0 & 0 & \mathbf{B_{4;5,6,7}} & B_{4;8,9} \\ 0 & 0 & 0 & B_{5;8,9} \\ 0 & 0 & 0 & B_{6;8,9} \\ 0 & 0 & 0 & \mathbf{B_{7;8,9}} \end{pmatrix}.$$

Lemma 3.9. *For each element of M there is a finite number of acceptable moves which brings the matrix to upper echelon form.*

Proof. We start from the first row and take acceptable moves to bring all highlighted entries in the first row in consecutive order. Since our goal is the upper echelon form, the updated highlighted entries will occupy $\mathbf{B_{1;k+1,\cdots,k+Q_1}}$ through $\mathbf{B_{1;k+Q_{j_1-1}+1,\cdots,k+Q_{j_1}}}$. Then if there are any highlighted entries on the second row, we bring them to positions $\mathbf{B_{2;k+Q_{j_1}+1,\cdots,k+Q_{j_1+1}}}$ through $\mathbf{B_{2;k+Q_{j_2-1}+1,\cdots,k+Q_{j_2}}}$. Here $j_1 < j_2$. Noticed that this will not effect the highlighted positions of the first row. If there is no highlighted entire on the second row, just leave it and move to the third row. Keep repeating these steps and we will end up with a special upper echelon matrix after finitely many steps. $\square$

Lemma 3.10. *Let $C_{k,n}$ be the number of $(k+Q_{n-1})\times n$ special upper echelon matrices of the type discussed above. Then $C_{k,n} \leq 2^{k+(p_0+1)(n-1)}$.*

Proof. The proof consists of two steps.

First, we disassemble the matrix by "lifting" all highlighted entries to the first row and put them in the same subsets if they were originally from the same row. In this way, the first row is partitioned into many subsets. Let P_n denote the number of all possible partitions, then

$$P_n = \sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{n-1}. \tag{3.7.19}$$

The idea is to put $n-1$ pads in the space among the n elements to separate them. Since we can separate them into different numbers (from 1 to n) of subsets, we can choose to use 0 pads, 1 pads, $\cdots$, upto $n-1$ pads. Hence (3.7.19) follows.

The second step is to reassemble the upper echelon matrix by "lowering" the first subset to the first used row, the second subset to the second used row, etc. Note here, we do not require that only the upper triangle matrix is used, which may result in more matrices. This does not matter since we are looking for an upper bound of the number of such matrices. Suppose an arbitrary partition of n has i subsets. Then there will be exactly $\binom{k+Q_{n-1}}{i}$ ways to lower them in an order preserving way to the $k+Q_{n-1}$ available rows. Thus

$$C_{k,n} \leq P_n \sum_{i=0}^{n} \binom{k+Q_{n-1}}{i} \leq 2^{k+Q_{n-1}+n-1} \leq 2^{k+(p_0+1)(n-1)}$$

as desired (since $Q_{n-1} = p_1 + p_2 + \cdots + p_{n-1} \leq (n-1)p_0$). $\square$

3.7.3 Equivalence classes

Let μ_s be a special upper echelon matrix. We write $\mu \sim \mu_s$ if μ can be transformed to μ_s in finitely many acceptable moves. Then we have the following corollary

Corollary 3.11. *There exists a subset D of $[0,t_k]^n$ such that*

$$\sum_{\mu\sim\mu_s} \int_0^{t_k} \dots \int_0^{t_{k+Q_{n-1}}} J^k(\underline{t}_{k+Q_n};\mu)dt_{k+Q_1}\dots dt_{k+Q_n} = \int_D J^k(\underline{t}_{k+Q_n};\mu_s)dt_{k+Q_1}\dots dt_{k+Q_n}. \tag{3.7.20}$$

Proof. Consider the following integral

$$I(\mu, id) = \int_0^{t_k} \dots \int_0^{t_{k+Q_{n-1}}} J^k(\underline{t}_{k+Q_n};\mu)dt_{k+Q_1}\dots dt_{k+Q_n}$$

and perform finitely many acceptable moves on the matrix associated to $I(\mu, id)$ until it is transformed to the special upper echelon matrix associated with $I(\mu_s,\sigma)$. By Lemma 3.8

$$I(\mu, id) = I(\mu_s, \sigma).$$

Assume that (μ_1, id) and (μ_2, id) with $\mu_1 \neq \mu_2$ yield the same echelon form μ_s, then the corresponding permutations σ_1 and σ_2 must be different. Therefore, D

can be chosen to be the union of all $\{t_k \geq t_{\sigma(k+Q_1)} \geq t_{\sigma(k+Q_2)} \geq \cdots \geq t_{\sigma(k+Q_n)}\}$ for all permutations σ which occur in a given equivalence class of some μ_s. $\square$

3.8 Proof of Theorem 3.7

Once the number of Duhamel term is reduced to a controllable size, we are ready to prove Theorem 3.7.

Proof of Theorem 3.7. Fix t_k. Recall the expansion of $\gamma^{(k)}$:

$$\gamma^{(k)}(t_k, \cdot) = \sum_{\mu \in M} \int_0^{t_k} \cdots \int_0^{t_{k+Q_{n-1}}} J^k(\underline{t}_{k+Q_n}; \mu) dt_{k+Q_1} \ldots dt_{k+Q_n} \tag{3.8.1}$$

and J^k:

$$\begin{aligned} J^k(\underline{t}_{k+Q_n}; \mu) &= e^{i(t_k - t_{k+Q_1})\Delta_{\pm}^{(k)}} B_{\mu(k+1);k+1,\cdots,k+Q_1} e^{i(t_{k+Q_1} - t_{k+Q_2})\Delta_{\pm}^{(k+Q_1)}} \cdots \\ &\times e^{i(t_{k+Q_{n-1}} - t_{k+Q_n})\Delta_{\pm}^{(k+Q_{n-1})}} B_{\mu(k+Q_{n-1}+1);k+Q_{n-1}+1,\cdots,k+Q_n}\left(\gamma^{(k+Q_n)}(t_{k+Q_n})\right). \end{aligned}$$

Thanks to Corollary 3.11 and Lemma 3.10 we can write $\gamma^{(k)}(t_k, \cdot)$ as a sum of at most $2^{k+(Q_1+1)(n-1)}$ terms of the form

$$\int_D J^k(\underline{t}_{k+Q_n}; \mu_s) dt_{k+Q_1} \ldots dt_{k+Q_n}. \tag{3.8.2}$$

Let $I_k^n = \overbrace{[0, t_k] \times [0, t_k] \times \cdots \times [0, t_k]}^{\text{n copies}}$
and $D_{t_{k+Q_1}} = \{(t_{k+Q_2}, \cdots, t_{k+Q_n}) | (t_{k+Q_1}, t_{k+Q_2}, \cdots, t_{k+Q_n}) \in D\}$, then

$$\left\| S^{(k,\alpha)} \int_0^{t_k} \cdots \int_0^{t_{k+Q_{n-1}}} J^k(\underline{t}_{k+Q_n}) dt_{k+Q_1} \ldots dt_{k+Q_n} \right\|_{L^2(\mathbb{R}^{2k} \times \mathbb{R}^{2k})}$$

$$\overset{\dagger}{\sim} \Big\| S^{(k,\alpha)} \int_D e^{i(t_k - t_{k+Q_1})\Delta_\pm^{(k)}} B_{\mu_s(k+1);k+1,\cdots,k+Q_1} e^{i(t_{k+Q_1} - t_{k+Q_2})\Delta_\pm^{(k+Q_1)}} \times B_{\mu_s(k+Q_1+1);k+Q_1+1,\cdots,k+Q_2} \cdots dt_{k+Q_1} \cdots dt_{k+Q_n} \Big\|_{L^2(\mathbb{R}^{2k}\times\mathbb{R}^{2k})}$$

$$\begin{aligned}
&= \Big\| \int_0^{t_k} e^{i(t_k - t_{k+Q_1})\Delta_\pm^{(k)}} \Big(\int_{D_{t_{k+Q_1}}} S^{(k,\alpha)} B_{\mu_s(k+1);k+1,\cdots,k+Q_1} e^{i(t_{k+Q_1} - t_{k+Q_2})\Delta_\pm^{(k+Q_1)}} \\
&\qquad \times B_{\mu_s(k+Q_1+1);k+Q_1+1,\cdots,k+Q_2} \cdots dt_{k+Q_2} \cdots dt_{k+Q_n} \Big) dt_{k+Q_1} \Big\|_{L^2(\mathbb{R}^{2k}\times\mathbb{R}^{2k})} \\
&\leq \int_0^{t_k} \Big\| e^{i(t_k - t_{k+Q_1})\Delta_\pm^{(k)}} \int_{D_{t_{k+Q_1}}} S^{(k,\alpha)} B_{\mu_s(k+1);k+1,\cdots,k+Q_1} e^{i(t_{k+Q_1} - t_{k+Q_2})\Delta_\pm^{(k+Q_1)}} \\
&\qquad \times B_{\mu_s(k+Q_1+1);k+Q_1+1,\cdots,k+Q_2} \cdots dt_{k+Q_2} \cdots dt_{k+Q_n} \Big\|_{L^2(\mathbb{R}^{2k}\times\mathbb{R}^{2k})} dt_{k+Q_1} \\
&= \int_0^{t_k} \Big\| \int_{D_{t_{k+Q_1}}} S^{(k,\alpha)} B_{\mu_s(k+1);k+1,\cdots,k+Q_1} e^{i(t_{k+Q_1} - t_{k+Q_2})\Delta_\pm^{(k+Q_1)}} \\
&\qquad \times B_{\mu_s(k+Q_1+1);k+Q_1+1,\cdots,k+Q_2} \cdots dt_{k+Q_2} \cdots dt_{k+Q_n} \Big\|_{L^2(\mathbb{R}^{2k}\times\mathbb{R}^{2k})} dt_{k+Q_1} \\
&\leq \int_0^{t_k} \Big(\int_{D_{t_{k+Q_1}}} \Big\| S^{(k,\alpha)} B_{\mu_s(k+1);k+1,\cdots,k+Q_1} e^{i(t_{k+Q_1} - t_{k+Q_2})\Delta_\pm^{(k+Q_1)}} \\
&\qquad \times B_{\mu_s(k+Q_1+1);k+Q_1+1,\cdots,k+Q_2} \cdots \Big\|_{L^2(\mathbb{R}^{2k}\times\mathbb{R}^{2k})} dt_{k+Q_2} \cdots dt_{k+Q_n} \Big) dt_{k+Q_1} \\
&\leq \int_{I_k^n} \Big\| S^{(k,\alpha)} B_{\mu_s(k+1);k+1,\cdots,k+Q_1} e^{i(t_{k+Q_1} - t_{k+Q_2})\Delta_\pm^{(k+Q_1)}} \\
&\qquad \times B_{\mu_s(k+Q_1+1);k+Q_1+1,\cdots,k+Q_2} \cdots \Big\|_{L^2(\mathbb{R}^{2k}\times\mathbb{R}^{2k})} dt_{k+Q_1} dt_{k+Q_2} \cdots dt_{k+Q_n} \\
&\overset{\text{C-S on } t_{k+Q_1}}{\leq} t_k^{\frac{1}{2}} \int_{I_k^{n-1}} \Big\| S^{(k,\alpha)} B_{\mu_s(k+1);k+1,\cdots,k+Q_1} e^{i(t_{k+Q_1} - t_{k+Q_2})\Delta_\pm^{(k+Q_1)}} \\
&\qquad \times \Big(B_{\mu_s(k+Q_1+1);k+Q_1+1,\cdots,k+Q_2} \cdots \Big) \Big\|_{L^2\left((t_{k+Q_1}\in[0,t_k])\times\mathbb{R}^{2k}\times\mathbb{R}^{2k}\right)} dt_{k+Q_2} \cdots dt_{k+Q_n}
\end{aligned}$$

† The implicit scalar is proportional to $2^{k+(p_0+1)(n-1)}$ which can be absorbed in $C^k(C_0\sqrt{T})^n$.

$$
\begin{aligned}
&\overset{(3.5.4)}{\leq} C_\alpha t_k^{\frac{1}{2}} \int_{I_k^{n-1}} \Big\| S^{(k+Q_1,\alpha)} B_{\mu_s(k+Q_1+1);k+Q_1+1,\cdots,k+Q_2} e^{i(t_{k+Q_2}-t_{k+Q_3})\Delta_{\pm}^{(k+Q_2)}} \\
&\qquad \times \Big(B_{\mu_s(k+Q_2+1);k+Q_2+1,\cdots,k+Q_3} \cdots \Big) \Big\|_{L^2\left(\mathbb{R}^{2(k+Q_1)}\times\mathbb{R}^{2(k+Q_1)}\right)} dt_{k+Q_2}\cdots dt_{k+Q_n} \\
&\leq \cdots \text{[repeatedly applying Cauchy-Schwartz inequality and (3.5.4)]} \\
&\leq (C_\alpha t_k^{\frac{1}{2}})^{n-1} \int_0^{t_k} \Big\| S^{(k+Q_{n-1},\alpha)} B_{\mu_s(k+Q_{n-1}+1);k+Q_{n-1}+1,\cdots,k+Q_n} \\
&\qquad\qquad \times \gamma^{k+Q_n}(t_{k+Q_n},\cdot) \Big\|_{L^2\left(\mathbb{R}^{2(k+Q_{n-1})}\times\mathbb{R}^{2(k+Q_{n-1})}\right)} dt_{k+Q_n} \\
&\overset{(3.4.6)}{\leq} (C_\alpha t_k^{\frac{1}{2}})^{n-1} C^{k+Q_n} \\
&\leq C^k (C_0\sqrt{T})^n.
\end{aligned}
$$

We choose appropriate C and C_0 to obtain the last line. As we have already seen in the proof of Theorem 3.6, the sum over $p_1, p_2, \cdots, p_n$ and product of potential constants $b_0^{(p_1)}, b_0^{(p_2)} \cdots, b_0^{(p_n)}$ will contribute an extra factor $p_0(b_0)^n$, which can be absorbed in constants C and C_0. This completes the proof. $\square$

Remark 3.12. *The main ingredients in the above proof are the free evolution bound* (3.5.4) *and a priori energy bound* (3.4.6)*. The a priori energy bound usually requires* $\alpha \leq 1$ *(may see* (3.4.17)*). While in* (3.5.4)*, we will need* $\alpha > \frac{d}{2} - \frac{1}{2(2p-1)}$ *which is at least* 1 *when* $d \geq 3$ *(see* (3.5.15)*). Therefore only the cases* $d = 1, 2$ *yield a nonempty intersection for the survival of* α*. Which implies that, under this setting, the method we used here to prove the uniqueness fails for the higher dimensional cases, unless we have better constrains on* α*. Klainerman and Machedon obtained a better estimate (on a different space) which allows them to prove the uniqueness for the case* $d = 3$*,* $p = 1$*. Actually, we are answering the same questions on the convergence of BBGKY* *hierarchy to p-GP hierarchy as in [32] (for* $d = 2$*,* $p = 1$*) and [7] (for* $d \leq 2$*,* $p = 2$*), for any positive integer* p*. The case when* $d = 3$*,* $p = 1$ *is covered by Chen-Pavlović [8] with a new approach.*

Chapter 4

Unconditional uniqueness of the cubic GP hierarchy at low regularity

Recently, Chen-Hainzl-Pavlović-Seiringer [5] presented a new, simpler proof of the unconditional uniqueness of solutions to the 3D cubic GP hierarchy, which is equivalent to the uniqueness result of Erdös-Schlein-Yau [15]. The authors employed the quantum de Finetti theorems (Theorem 4.2 and 4.3) combined with the Erdös-Schlein-Yau combinatorial method [14, 15, 16, 17] in the board game formulation as presented by Klainerman-Machedon [34]. The de Finetti theorems, which to some extend provides a factorized expression for the marginal density, is the main novelty that enables [5] to simplify the unconditional proof in [15]. In this chapter we present a joint work with Y. Hong and K. Taliaferro, in which we generalized the work of [5] and proved unconditional uniqueness for the cubic GP in Sobolev spaces with lower‡ regularity.

‡For the exact regularity index, please see the statement of Theorem 4.1

4.1 Cubic GP hierarchy revisited

As in previous chapters, we use $\mathbf{x}_k$ to denote the d-dimensional k-spatial variables $(x_1, x_2, ..., x_k)$. The corresponding Laplace operator is defined by $\Delta_{\mathbf{x}_k} = \sum_{j=1}^{k} \Delta_{x_j}$, and similarly for the primed variables. For each $k \in \mathbb{N}$, $\gamma^{(k)}$ is a bosonic density matrix on $L^2_{sym}(\mathbb{R}^{dk} \times \mathbb{R}^{dk})$ which is hermitian,

$$\gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) = \overline{\gamma}^{(k)}(t, \mathbf{x}'_k, \mathbf{x}_k),$$

and is symmetric in all components of $\mathbf{x}_k$, and in all components of $\mathbf{x}'_k$, respectively,

$$\gamma^{(k)}(t, x_{\sigma(1)}, \cdots, x_{\sigma(k)}, x'_{\sigma'(1)}, \cdots, x'_{\sigma'(k)}) = \gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k)$$

for any permutations σ, σ' on k elements.

Recall that the cubic GP hierarchy in $\mathbb{R}^d$ as an infinite system of coupled linear equations given by (see §1.1.1 or §3.1.2 for the definition of general GP hierarchy)

$$i\partial_t \gamma^{(k)} = (-\Delta_{\mathbf{x}_k} + \Delta_{\mathbf{x}'_k})\gamma^{(k)} + \lambda B_{k+1}\gamma^{(k+1)}, \quad \forall k \in \mathbb{N}, \tag{4.1.1}$$

where $\gamma^{(k)} = \gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) : I \times \mathbb{R}^{dk} \times \mathbb{R}^{dk} \to \mathbb{C}$, $I \subset \mathbb{R}$ is a time interval and $\lambda = \pm 1$.

Here the contraction operator is defined as

$$B_{k+1} = \sum_{j=1}^{k} B_{j;k+1} = \sum_{j=1}^{k} (B^+_{j;k+1} - B^-_{j;k+1})$$

with

$$\Big(B^+_{j;k+1}\gamma^{(k+1)}\Big)(t, \mathbf{x}_k, \mathbf{x}'_k)$$

$$= \int dx_{k+1} dx'_{k+1} \delta(x_j - x_{k+1}) \delta(x_j - x'_{k+1}) \gamma^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}'_{k+1})$$

and each $B^-_{j;k+1}$ contracts the triple x'_j, x_{k+1}, x'_{k+1},

$$\begin{aligned}&\left(B^-_{j;k+1}\gamma^{(k+1)}\right)(t, \mathbf{x}_k, \mathbf{x}'_k)\\ &= \int dx_{k+1} dx'_{k+1} \delta(x'_j - x_{k+1}) \delta(x'_j - x'_{k+1}) \gamma^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}'_{k+1}).\end{aligned}$$

The cubic GP hierarchy is called *focusing* (*defocusing*, respectively) if $\lambda = 1$ ($\lambda = -1$, respectively).

We will call the uniqueness of solutions to the GP hierarchy *unconditional* if it holds without assuming any a priori bound of the form (1.1.11).

4.2 Statement of the main result

Before we state the main theorem, let us recall the relevant notation.

Let $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ be a sequence of bosonic density matrices on $L^2_{sym}(\mathbb{R}^{dk} \times \mathbb{R}^{dk})$. We say that $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ is *admissible* if $\gamma^{(k)}$ is a non-negative trace class operator on $L^2_{sym}(\mathbb{R}^{dk} \times \mathbb{R}^{dk})$ and $\gamma^{(k)} = \mathrm{Tr}_{k+1}(\gamma^{(k+1)})$ for all $k \in \mathbb{N}$. We call a sequence $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ a *limiting hierarchy* if there is a sequence $\{\gamma^{(k)}_N\}_{N\in\mathbb{N}:N\geq k}$ of non-negative density matrices on $L^2_{sym}(\mathbb{R}^{dN})$ with $\mathrm{Tr}(\gamma^{(k)}_N) = 1$ such that $\gamma^{(k)}$ is the weak* limit of the k-particle marginals of $\gamma^{(k)}_N$ in the trace class on $L^2_{sym}(\mathbb{R}^{dk} \times \mathbb{R}^{dk})$, that is,

$$\gamma^{(k)}_N := \mathrm{Tr}_{k+1,\cdots,N}(\gamma^{(N)}_N) \rightharpoonup^* \gamma^{(k)} \text{ as } N \to \infty.$$

For $s \in \mathbb{R}$, we define the function space $\mathfrak{H}^s$ as the collection of sequences $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ of density matrices on $L^2_{sym}(\mathbb{R}^{dk}\times\mathbb{R}^{dk})$ such that

$$\operatorname{Tr}(|S^{(k,s)}\gamma^{(k)}|) < M^{2k} \quad \forall k \in \mathbb{N} \text{ for some constant } M > 0,$$

where $S^{(k,s)}$ is defined as in (3.4.1),

$$S^{(k,s)} := \prod_{j=1}^{k}(1-\Delta_{x_j})^{\frac{s}{2}}(1-\Delta_{x_j'})^{\frac{s}{2}}.$$

We say that $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ is a *mild solution*, in the space $L^\infty_{t\in[0,T)}\mathfrak{H}^s$, to the cubic GP hierarchy (4.1.1) with initial data $\{\gamma^{(k)}(0)\}_{k\in\mathbb{N}}$ if it solves the integral equation

$$\gamma^{(k)}(t) = U^{(k)}(t)\gamma^{(k)}(0) + i\lambda\int_0^t U^{(k)}(t-s)B_{k+1}\gamma^{(k+1)}(s)ds,$$

where $U^{(k)}(t) := e^{it(\Delta_{\mathbf{x}_k}-\Delta_{\mathbf{x}_k'})}$, and satisfies the bound

$$\sup_{t\in[0,T)}\operatorname{Tr}(|S^{(k,s)}\gamma^{(k)}(t)|) < M^{2k} \quad \forall k \in \mathbb{N} \text{ for some constant } M > 0.$$

Our main theorem states that any mild solution to the cubic GP hierarchy, which is either admissible or a limiting hierarchy, is unconditionally unique in $L^\infty_{t\in[0,T)}\mathfrak{H}^s$ for a certain range of s. More precisely:

Theorem 4.1 (Unconditional uniqueness)**.** *Let*

$$\begin{cases} s \geq \frac{d}{6} & \text{if } d = 1, 2, \\ s > s_c & \text{if } d \geq 3, \end{cases} \tag{4.2.1}$$

where $s_c = \frac{d-2}{2}$*. If* $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ *is a mild solution in* $L^\infty_{t\in[0,T)}\mathfrak{H}^s$ *to the focusing (defocusing, respectively) cubic GP hierarchy* (4.1.1) *with initial data*

$\{\gamma^{(k)}(0)\}_{k\in\mathbb{N}}$, *which is either admissible or a limiting hierarchy for each* t, *then it is the only such solution for the given initial data.*

Our theorem reduces the regularity requirement for unconditional uniqueness for the GP hierarchy in [5]. We remark that the regularity assumption in (4.2.1) is the same as in the currently known unconditional uniqueness results for the cubic NLS

$$i\partial_t\phi + \Delta\phi - \lambda|\phi|^2\phi = 0,\ \phi(0) = \phi_0 \in H^s. \tag{4.2.2}$$

For NLS, by *unconditional uniqueness*, we mean uniqueness of solutions in the Sobolev space H^s itself, while uniqueness in the intersection of the Sobolev space and auxiliary spaces is called conditional. By the contraction mapping argument with auxiliary Strichartz spaces, the conditional uniqueness is proved for the NLS (4.2.2) in H^s with $s \geq \max(s_c, 0)$, where $s_c = \frac{d-2}{2}$ (see [3]). However, the unconditional uniqueness is proved in H^s only for s in (4.2.1), and it is an open problem to reduce s in one and two dimensions [30, 19, 46, 55, 27].

Our proof uses the Klainerman-Machedon board game formulation [34] of the combinatorial argument of Erdös-Schlein-Yau [14, 15, 16, 17], and the method of Chen-Hainzl-Pavlović-Seiringer [5] via the quantum de Finetti theorem (see §4.3.1). The crucial advantage of using the quantum de Finetti theorem is that it provides a factorized representation of solutions to the GP hierarchy in the integral form (see (4.4.4)). This structure allows us to make use of techniques of NLS theory to analyze solutions to the GP hierarchies (see [5] and [4]).

As described in Section §6.1.1 of [5], the main difficulty in lowering regularity requirement comes from the last cubic term $\||\phi|^2\phi\|_{L^2} = \|\phi\|_{L^6}^3$ in the so called *distinguished tree*. Indeed, this last term can be controlled by the Sobolev norm $\|\phi\|_{H^s}^3$ only for $s \geq 1$ in $\mathbb{R}^3$, as it was done in [5]. We address the issue of lowering regularity s by using the dispersive estimate

$$\|e^{it\Delta}f\|_{L^{\frac{6}{1+2\epsilon}}} \lesssim |t|^{-(1-\epsilon)}\|f\|_{L^{\frac{6}{5-2\epsilon}}}$$

in $\mathbb{R}^3$, for instance. Indeed, if one applies the dispersive estimate and the endpoint Strichartz estimate to the factorized representation of the solution in the framework of [5], one gets a better last cubic term $\||\phi|^2\phi\|_{L^{\frac{6}{5-2\epsilon}}} = \|\phi\|^3_{L^{\frac{18}{5-2\epsilon}}}$, which allows us to reduce s to $\frac{2}{3}+\epsilon$. The regularity requirement in the classical Kato's work on the unconditional uniqueness for the 3D cubic NLS [30] can be obtained in this way. We further reduce s almost down to the critical regularity by employing negative order Sobolev spaces (Lemma 4.2), which are well-known tools in the literature on unconditional uniqueness for NLS. Combining the dispersive estimate, the Strichartz estimates and negative Sobolev norms, we formulate the key trilinear estimates (Lemma 4.1) in our proof.

4.3 Main tools

In the proof of Theorem 4.1, we use the Strong and Weak de Finetti theorems and trilineaer estimates.

4.3.1 Strong and weak de Finetti theorems

The quantum de Finetti theorem is one of the two key tools in our proof. It is a quantum analogue of the Hewitt-Savage theorem in probability theory. We recall two versions of the de Finetti theorem, a strong version and a weak version.

In Hudson-Moody [29] and Stormer [51], the theorem applies to a sequence of density matrices satisfying the *admissibility*: $\gamma^{(k)} = \mathrm{Tr}(\gamma^{(k+1)})$ for all $k \in \mathbb{N}$. We state the strong de Finetti theorem in the formulation of Lewin-Nam-Rougerie [38].

Theorem 4.2 (Strong quantum de Finetti theorem [29, 51, 38])**.** *If a sequence $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ of bosonic density matrices on $L^2_{sym}(\mathbb{R}^{dk} \times \mathbb{R}^{dk})$ is admissible, then there exists a unique Borel probability measure μ, supported on the unit sphere $S \subset L^2(\mathbb{R}^d)$ and invariant under multiplication of $\phi \in L^2(\mathbb{R}^d)$ by complex numbers of modulus one, such that*

$$\gamma^{(k)} = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \quad k \in \mathbb{N}. \tag{4.3.1}$$

In the works [14, 15, 16, 17], the density matrices $\gamma^{(k)}$ obtained as the weak* limit of corresponding density matrices of BBGKY do not necessarily satisfy the admissibility. However, in this context, we still have a similar conclusion to that of Theorem 4.2, thanks to the weak de Finetti theorem. This version is due to Ammari and Nier [1, 2] and Lewin-Nam-Rougerie [38]. We stated as it is formulated in [38].

Theorem 4.3 (Weak quantum de Finetti theorem [1, 2, 38]). *If a sequence $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ of bosonic density matrices on $L^2_{sym}(\mathbb{R}^{dk}\times\mathbb{R}^{dk})$ is a limiting hierarchy, then there exists a unique Borel probability measure μ, supported on the unit ball $\mathcal{B}\subset L^2(\mathbb{R}^d)$ and invariant under multiplication of $\phi\in L^2(\mathbb{R}^d)$ by complex numbers of modulus one, such that (4.3.1) holds.*

4.3.2 Trilinear estimates

Our proof relies on the following lemma in a crucial way.

Lemma 4.1 (Trilinear estimates). *We define the trilinear form T by*

$$T(f,g,h)=(e^{i(t-t_1)\Delta}f)(e^{i(t-t_2)\Delta}g)(e^{i(t-t_3)\Delta}h).$$

(i) $d\geq 3$. For small $\epsilon>0$, we have

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)}W_x^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}\lesssim T^\epsilon\|f\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}\|g\|_{H^{s_\epsilon}}\|h\|_{H^{s_\epsilon}}, \tag{4.3.2}$$

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)}H_x^{s_\epsilon}}\lesssim T^\epsilon\|f\|_{H^{s_\epsilon}}\|g\|_{H^{s_\epsilon}}\|h\|_{H^{s_\epsilon}}, \tag{4.3.3}$$

where $s_\epsilon=s_c+\epsilon=\frac{d-2}{2}+\epsilon$, $r_\epsilon=\frac{2d}{d+2(1-\epsilon)}$.
(ii) $d=2$. For small $\epsilon>0$, we have

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)}W_x^{-(\frac{1}{3}-\frac{\epsilon}{2}),\frac{2}{2-\epsilon}}}\lesssim T^\epsilon\|f\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}),\frac{2}{2-\epsilon}}}\|g\|_{H^{1/3}}\|h\|_{H^{1/3}}, \tag{4.3.4}$$

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)}H_x^{1/3}}\lesssim T^{1/3}\|f\|_{H^{1/3}}\|g\|_{H^{1/3}}\|h\|_{H^{1/3}}. \tag{4.3.5}$$

(iii) $d=1$. We have

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)}L^1_x}\lesssim T^{1/2}\|f\|_{L^1}\|g\|_{L^2}\|h\|_{L^2}, \tag{4.3.6}$$

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)}L^2_x}\lesssim T^{1/2}\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}. \tag{4.3.7}$$

We will prove Lemma 4.1 using the dispersive estimate, the Strichartz estimates and negative order Sobolev norms.

The following lemma gives a version of the product rule in the context of negative order Sobolev spaces.

Lemma 4.2 (Negative order Sobolev spaces)**.** *Let $\epsilon > 0$ be a small number. Then, for $s \geq s_c + \frac{\epsilon}{2}$, we have*

$$\|fg\|_{W^{-s,r_\epsilon}} \lesssim \|f\|_{W^{-s,r'_\epsilon}} \|g\|_{W^{s,\frac{2d}{d+2-3\epsilon}}},$$

where $r_\epsilon = \frac{2d}{d+2(1-\epsilon)}$.

Before we give a proof of Lemma 4.2 we recall the fractional Leibniz rule according to [26, 37] that will be used in the proof of the lemma. The generalized Leibniz rule is proved in [26] for Riesz and Bessel potentials of order $s \in \mathbb{R}$. We record the version for Bessel potential here:

Lemma 4.3 (Fractional Leibniz rule)**.** *Suppose that $1 < p < \infty$, $s \geq 0$, and $\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{p}$ with $i = 1, 2$, $1 < p_i \leq \infty$, $1 \leq q_i < \infty$. Then*

$$\|fg\|_{W^{s,p}} \lesssim \|f\|_{W^{s,p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|g\|_{W^{s,q_2}}. \tag{4.3.8}$$

Proof of Lemma 4.2. By Hölder's inequality, the fractional Leibniz rule and the Sobolev inequality, we have

$$\begin{aligned}\left| \int f(x) g(x) \overline{h}(x) dx \right| &\leq \|f\|_{W^{-s,r'_\epsilon}} \|g\overline{h}\|_{W^{s,r_\epsilon}} \\ &\lesssim \|f\|_{W^{-s,r'_\epsilon}} \Big(\|g\|_{W^{s,\frac{2d}{d+2-3\epsilon}}} \|h\|_{L^{\frac{2d}{\epsilon}}} + \|g\|_{L^{\frac{d}{2(1-\epsilon)}}} \|h\|_{W^{s,r'_\epsilon}} \Big)\end{aligned}$$

$$\lesssim \|f\|_{W^{-s,r'_\epsilon}} \|g\|_{W^{s,\frac{2d}{d+2-3\epsilon}}} \|h\|_{W^{s,r'_\epsilon}}.$$

The lemma now follows from the standard duality argument. $\square$

Now we are ready to present a proof of Lemma 4.1.

Proof of Lemma 4.1. (i). For notational convenience, we omit the time interval $[0,T)$ in the norms.

Towards (4.3.2): By Lemma 4.2, we obtain

$$\begin{aligned}
&\|T(f,g,h)\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \\
&\lesssim \|e^{i(t-t_1)\Delta}f\|_{W^{-(s_c+\frac{\epsilon}{2}),r'_\epsilon}} \|(e^{i(t-t_2)\Delta}g)(e^{i(t-t_3)\Delta}h)\|_{W^{s_c+\frac{\epsilon}{2},\frac{2d}{d+2-3\epsilon}}} \\
&\lesssim \tfrac{1}{|t-t_1|^{1-\epsilon}} \|f\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|g\|_{H^{s_\epsilon}} \|h\|_{H^{s_\epsilon}}.
\end{aligned} \tag{4.3.9}$$

Here, in the second inequality, we use the dispersive estimate:

$$\|e^{i(t-t_1)\Delta}f\|_{W^{-(s_c+\frac{\epsilon}{2}),r'_\epsilon}} \lesssim \tfrac{1}{|t-t_1|^{1-\epsilon}} \|f\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}$$

and the fractional Leibniz rule and the Sobolev inequality:

$$\begin{aligned}
&\|(e^{i(t-t_2)\Delta}g)(e^{i(t-t_3)\Delta}h)\|_{W^{s_c+\frac{\epsilon}{2},\frac{2d}{d+2-3\epsilon}}} \\
&\lesssim \|e^{i(t-t_2)\Delta}g\|_{W^{s_c+\frac{\epsilon}{2},\frac{2d}{d-\epsilon}}} \|e^{i(t-t_3)\Delta}h\|_{L^{\frac{d}{1-\epsilon}}} + \|e^{i(t-t_2)\Delta}g\|_{L^{\frac{d}{1-\epsilon}}} \|e^{i(t-t_3)\Delta}h\|_{W^{s_c+\frac{\epsilon}{2},\frac{2d}{d-\epsilon}}} \\
&\lesssim \|e^{i(t-t_2)\Delta}g\|_{H^{s_\epsilon}} \|e^{i(t-t_3)\Delta}h\|_{H^{s_\epsilon}} = \|g\|_{H^{s_\epsilon}} \|h\|_{H^{s_\epsilon}}.
\end{aligned} \tag{4.3.10}$$

Integrating out the time variable t, we prove (4.3.2).

Towards (4.3.3): By the fractional Leibniz rule, we have

$$\|T(f,g,h)\|_{L^1_t H^{s_\epsilon}_x} \lesssim \|e^{i(t-t_1)\Delta}f\|_{L^3_t W_x^{s_\epsilon,\frac{6d}{3d-4}}} \|e^{i(t-t_2)\Delta}g\|_{L^3_t L^{3d}_x} \|e^{i(t-t_3)\Delta}h\|_{L^3_t L^{3d}_x}$$

$$+ \|e^{i(t-t_1)\Delta} f\|_{L^3_t L^{3d}_x} \|e^{i(t-t_2)\Delta} g\|_{L^3_t W^{s_\epsilon, \frac{6d}{3d-4}}_x} \|e^{i(t-t_3)\Delta} h\|_{L^3_t W^{3d}_x}$$
$$+ \|e^{i(t-t_1)\Delta} f\|_{L^3_t L^{3d}_x} \|e^{i(t-t_2)\Delta} g\|_{L^3_t L^{3d}_x} \|e^{i(t-t_3)\Delta} h\|_{L^3_t W^{s_\epsilon, \frac{6d}{3d-4}}_x}.$$

Then, by the Sobolev inequality and the Strichartz estimates, we bound the first term by

$$\begin{aligned}
&\lesssim \|e^{i(t-t_1)\Delta} f\|_{L^3_t W^{s_\epsilon, \frac{6d}{3d-4}}_x} \|e^{i(t-t_2)\Delta} g\|_{L^3_t W^{s_\epsilon, \frac{6d}{3d-4+6\epsilon}}_x} \|e^{i(t-t_3)\Delta} h\|_{L^3_t W^{s_\epsilon, \frac{6d}{3d-4+6\epsilon}}_x} \\
&\leq T^\epsilon \|e^{i(t-t_1)\Delta} f\|_{L^3_t W^{s_\epsilon, \frac{6d}{3d-4}}_x} \|e^{i(t-t_2)\Delta} g\|_{L^{\frac{6}{2-3\epsilon}}_t W^{s_\epsilon, \frac{6d}{3d-4+6\epsilon}}_x} \|e^{i(t-t_3)\Delta} h\|_{L^{\frac{6}{2-3\epsilon}}_t W^{s_\epsilon, \frac{6d}{3d-4+6\epsilon}}_x} \\
&\lesssim T^\epsilon \|f\|_{H^{s_\epsilon}} \|g\|_{H^{s_\epsilon}} \|h\|_{H^{s_\epsilon}}.
\end{aligned}$$

Similarly, we bound the other two terms.

(ii). Towards (4.3.4): The proof is similar to that of (4.3.2), but here we use Lemma 4.2 with $s = \frac{1}{3} - \frac{\epsilon}{2}$. Indeed, by Lemma 4.2 and the dispersive estimate,

$$\begin{aligned}
\|T(f,g,h)\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}), r_\epsilon}} &\lesssim \|e^{i(t-t_1)\Delta} f\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}), r'_\epsilon}} \|(e^{i(t-t_2)\Delta} g)(e^{i(t-t_3)\Delta} h)\|_{W^{\frac{1}{3}-\frac{\epsilon}{2}, \frac{2d}{d+2-3\epsilon}}} \\
&\lesssim \tfrac{1}{|t-t_1|^{1-\epsilon}} \|(e^{i(t-t_2)\Delta} g)(e^{i(t-t_3)\Delta} h)\|_{W^{\frac{1}{3}-\frac{\epsilon}{2}, \frac{2d}{d+2-3\epsilon}}}.
\end{aligned}$$

Then apply (4.3.8) and Sobolev inequality, we obtain

$$\begin{aligned}
&\|(e^{i(t-t_2)\Delta} g)(e^{i(t-t_3)\Delta} h)\|_{W^{\frac{1}{3}-\frac{\epsilon}{2}, \frac{2d}{d+2-3\epsilon}}} \\
&\lesssim \|e^{i(t-t_2)\Delta} g\|_{W^{\frac{1}{3}-\frac{\epsilon}{2}, \frac{2d}{d-\epsilon}}} \|e^{i(t-t_3)\Delta} h\|_{L^{\frac{d}{1-\epsilon}}} + \|e^{i(t-t_2)\Delta} g\|_{L^{\frac{d}{1-\epsilon}}} \|e^{i(t-t_3)\Delta} h\|_{W^{\frac{1}{3}-\frac{\epsilon}{2}, \frac{2d}{d-\epsilon}}} \\
&\lesssim \|e^{i(t-t_2)\Delta} g\|_{H^{1/3}} \|e^{i(t-t_3)\Delta} h\|_{H^{1/3}} = \|g\|_{H^{1/3}} \|h\|_{H^{1/3}},
\end{aligned}$$

Applying this to the above inequality and integrating out t, we complete the proof.

Towards (4.3.5): Although we set ϵ to be small and $d \geq 3$ in the proof of (4.3.3), the same argument actually works for $\epsilon = \frac{1}{3}$ and $d = 2$ which is exactly (4.3.5).

(iii). For (4.3.6), by the Hölder inequality and the 1D dispersive estimates, we have

$$\|T(f,g,h)\|_{L^1} \leq \|e^{i(t-t_1)}f\|_{L^\infty}\|e^{i(t-t_2)}g\|_{L^2}\|e^{i(t-t_3)}h\|_{L^2} \lesssim \frac{1}{|t-t_1|^{1/2}}\|f\|_{L^1}\|g\|_{L^2}\|h\|_{L^2}.$$

Integrating out the time variable t, we prove (4.3.6).

For (4.3.7), by the Hölder inequality and the Strichartz estimate,

$$\begin{aligned}\|T(f,g,h)\|_{L_t^1 L_x^2} &\leq T^{1/2}\|e^{i(t-t_1)}f\|_{L^6_{t,x}}\|e^{i(t-t_2)}g\|_{L^6_{t,x}}\|e^{i(t-t_3)}h\|_{L^6_{t,x}} \\ &\lesssim T^{1/2}\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}.\end{aligned}$$

□

4.4 The proof of the main theorem

In this section, we prove the main theorem. First, in §4.4.1, we present the setup of the proof. In §4.4.2, we reduce the proof of the main theorem to the key lemma (Lemma 4.5), via the quantum de Finetti theorem. The rest of the paper is then devoted to the proof of Lemma 4.5.

4.4.1 Set up of the proof

The setup of the proof is similar to that of Chen-Hainzl-Pavlović-Seiringer [5], but we use a negative order Sobolev type norm to lower the regularity.

Let $\{\gamma_1^{(k)}(t)\}_{k\in\mathbb{N}}$ and $\{\gamma_2^{(k)}(t)\}_{k\in\mathbb{N}}$ be two mild solutions in $L^\infty_{t\in[0,T)}\mathfrak{H}^s$ to the cubic GP hierarchy with the same initial data, which are either admissible or limiting hierarchies. For uniqueness, it is enough to show that their difference $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$, given by

$$\gamma^{(k)}(t) := \gamma_1^{(k)}(t) - \gamma_2^{(k)}(t), \qquad k\in\mathbb{N},$$

vanishes for all k in a certain norm.

Due to the linearity of the GP hierarchy, it follows that the difference $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ solves the GP hierarchy with zero initial data. Hence, each $\gamma^{(k)}(t)$ satisfies the integral equation

$$\gamma^{(k)}(t) = i\lambda\int_0^t U^{(k)}(t-t_1)B_{k+1}\gamma^{(k+1)}(t_1)dt_1.$$

Now fix k. Iterating this integral equation $(n-1)$ times, we write

$$\begin{aligned}\gamma^{(k)}(t) = (i\lambda)^n\int_{t_n\leq\cdots\leq t_1\leq t} & U^{(k)}(t-t_1)B_{k+1}\cdots\\ &\times\cdots U^{(k+n-1)}(t_{n-1}-t_n)B_{k+n}\gamma^{(k+n)}(t_n)dt_1\cdots dt_n.\end{aligned}$$

For notational convenience, we denote $(k+1)$-temporal variables $(t_0,t_1,\cdots,t_n)$ by $\underline{t}_n$ with $t_0=t$, and the linear propagator $U^{(i)}(t_j-t_{j'})$ by $U^{(i)}_{j,j'}$. Then, we rewrite $\gamma^{(k)}(t)$ in a compact form as

$$\gamma^{(k)}(t) = (i\lambda)^n\int_{t_n\leq\cdots\leq t_1\leq t} J^k(\underline{t}_n)d\underline{t}_n, \tag{4.4.1}$$

where

$$J^k(\underline{t}_n) := U^{(k)}_{0,1}B_{k+1}U^{(k)}_{1,2}B_{k+2}\cdots U^{(k+n-1)}_{n-1,n}B_{k+n}\gamma^{(k+n)}(t_n).$$

By density, our uniqueness theorem follows from uniqueness in an even weaker norm.

Proposition 4.4. *For all $t \in [0, T)$ with $T > 0$ small enough, the trace norm of $S^{(k,-d)}$ applied to* (4.4.1) *vanishes as $n \to \infty$ uniformly in k, that is*

$$\operatorname{Tr}(|S^{(k,-d)}\gamma^{(k)}(t)|) = 0, \quad \forall k, \tag{4.4.2}$$

where $d > 0$ is the dimension.

By applying the combinatorial method of [34], that was presented in chapter 2, we rewrite $\gamma^{(k)}$ as follows.

$$\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{M}_{k,n}} \int_{D_{\sigma,t}} d\underline{t}_n J^k(\underline{t}_n; \sigma), \tag{4.4.3}$$

where we used the notation of chapter 2.

4.4.2 Proof of the main theorem

As mentioned above, it suffices to show Proposition 4.4. For the proof, we use the framework of Chen-Hainzl-Pavlović-Seiringer [5] via the quantum de Finetti theorem.

Applying the strong (4.2) or the weak (4.3) quantum de Finetti theorem, we write

$$\gamma^{(k)}(t) = \int d\tilde{\mu}_t(\phi)(|\phi\rangle\langle\phi|)^{\otimes k}, \quad \forall k \in \mathbb{N}, \tag{4.4.4}$$

where $\tilde{\mu}_t = \mu_t^{(1)} - \mu_t^{(2)}$ with

$$\gamma_i^{(k)}(t) = \int d\mu_t^{(i)}(\phi)(|\phi\rangle\langle\phi|)^{\otimes k}, \quad i = 1, 2.$$

Plugging (4.4.4) into $J^k(\underline{t}_n; \sigma)$ in the reduced Duhamel expansion (4.4.3), we obtain a new expression

$$\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{M}_{k,n}} \int_{D_{\sigma,t}} d\underline{t}_n \int d\tilde{\mu}_{t_n}(\phi) J^k(\underline{t}_n; \sigma), \tag{4.4.5}$$

where

$$J^k(\underline{t}_n; \sigma) = U_{0,1}^{(k)} B_{\sigma(k+1);k+1} U_{1,2}^{(k+1)} B_{\sigma(k+2);k+2} \cdots U_{n-1,n}^{(k+n-1)} B_{\sigma(k+n);k+n} (|\phi\rangle\langle\phi|)^{\otimes(k+n)}. \tag{4.4.6}$$

Then, we formulate the following key lemma that implies Proposition 4.4 (and thus the main theorem).

Lemma 4.5 (Key lemma). *There exists a uniform constant $C > 0$ such that for arbitrarily small $\epsilon > 0$, we have*

$$\int_{[0,T)^{n-1}} d\underline{t}_{n-1} \mathrm{Tr}(|S^{(k,-d)} J^k(\underline{t}_n; \sigma)|) \leq \begin{cases} (CT^\epsilon)^{n-1} \|\phi\|_{H^{s_\epsilon}}^{2(k+n)} & \text{if } d \geq 3 \\ (CT^{1/3})^{n-1} \|\phi\|_{H^{1/3}}^{2(k+n)} & \text{if } d = 2 \\ (CT^{1/2})^{n-1} \|\phi\|_{H^{1/6}}^{2(k+n)} & \text{if } d = 1, \end{cases} \tag{4.4.7}$$

where $s_\epsilon = \frac{d-2}{2} + \epsilon$.

Proof of Theorem 4.1, assuming Lemma 4.5. We present the proof for the case $d \geq 3$ only. Indeed, when $d = 1$ ($d = 2$, resp), it can be proved in an analogous way by replacing the H^{s_c} norm with the $H^{1/6}$ norm (the $H^{1/3}$ norm, resp).

Let $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ be as above. The goal is to show $\mathrm{Tr}(|S^{(k,-d)}\gamma^{(k)}(t)|) = 0$ for all $k \in \mathbb{N}$. Applying the triangle inequality and Lemma 4.5, we write

$$\begin{aligned}\mathrm{Tr}(|S^{(k,-d)}\gamma^{(k)}(t)|) &\leq \sum_{i=1,2}\sum_{\sigma\in\mathcal{M}_{k,n}}\int_{[0,T)^n} d\underline{t}_n \int d\mu_{t_n}^{(i)}(\phi)\mathrm{Tr}(|S^{(k,-d)}J^k(\underline{t}_n;\sigma)|)\\ &\leq (CT^\epsilon)^{n-1}T\sum_{i=1,2}\sum_{\sigma\in\mathcal{M}_{k,n}}\sup_{t_n\in[0,T)}\int d\mu_{t_n}^{(i)}(\phi)\|\phi\|_{H^{s_\epsilon}}^{2(k+n)}.\end{aligned} \tag{4.4.8}$$

We claim that there exists $M > 0$ such that

$$\|\phi\|_{H^{s_\epsilon}} \leq M \quad \text{a.s. } \mu_t^{(i)}, \quad \forall t \in [0,T). \tag{4.4.9}$$

Indeed, since $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}} \in L^\infty_{t\in[0,T)}\mathfrak{H}^s$, there exists $M > 0$ such that

$$\int d\mu_t^{(i)}(\phi)\|\phi\|_{H^s}^{2k} = \mathrm{Tr}(|S^{(k,s)}\gamma^{(k)}(t)|) < M^{2k}, \quad \forall k \in \mathbb{N}. \tag{4.4.10}$$

Hence, it follows from the Chebyshev inequality that for $\lambda > M$,

$$\mu_t^{(i)}\big(\{\phi \in L^2 : \|\phi\|_{H^s} > \lambda\}\big) \leq \frac{1}{\lambda^{2k}}\int d\mu_t^{(i)}(\phi)\|\phi\|_{H^s}^{2k} < \Big(\frac{M}{\lambda}\Big)^{2k} \to 0 \quad \text{as } k\to\infty. \tag{4.4.11}$$

Returning to (4.4.8), by (4.4.9) and Lemma 2.4, we prove that

$$\begin{aligned}\mathrm{Tr}(|S^{(k,-d)}\gamma^{(k)}(t)|) &\leq (CT^\epsilon)^{n-1}T\cdot 2\cdot 2^{k+2n-2}\cdot M^{2(k+n)}\\ &= \frac{M^{2k}2^{k-1}T}{CT^\epsilon}(4CT^\epsilon M^2)^n \to 0 \text{ as } n\to\infty.\end{aligned} \tag{4.4.12}$$

for $T < (4CM^2)^{-1/\epsilon}$. □

The remainder of this chapter will be devoted to proving Lemma 4.5. We remark that our proof heavily relies on the above trilinear estimates 4.1.

We will prove Lemma 4.5 in the following sections. To this end, we proceed as in [5] and use *binary tree graphs* to help organize the terms in $J^k(\underline{t}_n, \sigma)$ (see (4.4.6)). For the reader's convenience, before proving the lemma, we give an example calculation in §4.5. The trilinear estimates in Lemma 4.1 are the key estimates, and will be applied recursively in general case (see §4.7).

4.5 An example

In this section, we illustrate the ideas of the proof of Lemma 4.5 via an example.

Let $d \geq 3$, $k = 2$ and $n = 4$ in Lemma 4.5. We investigate the example

$$\int_{[0,T)^3} d\underline{t}_3 \mathrm{Tr}(|S^{(2,-d)} J^2(\underline{t}_4; \sigma)|) \tag{4.5.1}$$

with a specific map σ represented by the matrix

$$\begin{pmatrix} \mathbf{B_{1;3}} & B_{1;4} & B_{1;5} & B_{1;6} \\ B_{2;3} & \mathbf{B_{2;4}} & B_{2;5} & B_{2;6} \\ 0 & B_{3;4} & \mathbf{B_{3;5}} & \mathbf{B_{3;6}} \\ 0 & 0 & B_{4;5} & B_{4;6} \end{pmatrix}. \tag{4.5.2}$$

In other words,

$$J^2 = J^2(\underline{t}_4; \sigma) = U^{(2)}_{0,1} B_{1;3} U^{(3)}_{1,2} B_{2;4} U^{(4)}_{2,3} B_{3;5} U^{(5)}_{3,4} B_{3;6} (|\phi\rangle \langle\phi|)^{\otimes 6}. \tag{4.5.3}$$

To this end, in §4.5.1-4.5.2, we organize the terms in $J^2(\underline{t}_4, \sigma)$. Then, in §4.5.3, we estimate the example by the trilinear estimates (Lemma 4.1).

4.5.1 Factorization of J^2

We will decompose J^2 into two one-particle density matrices by examining the effect of the contraction operators starting with the last one on the RHS of (4.5.3). We denote each factor in the last term $(|\phi\rangle\langle\phi|)^{\otimes 6}$ by u_i, ordered by increasing index i, so that $(|\phi\rangle\langle\phi|)^{\otimes 6} = \otimes_{i=1}^{6} u_i$.

First of all, in (4.5.3), the last interaction operator $B_{3;6}$ contracts the factor u_3 and u_6, and leaves all other factors unchanged,

$$B_{3;6}(\otimes_{i=1}^{6} u_i) = u_1 \otimes u_2 \otimes \Theta_4 \otimes u_4 \otimes u_5. \tag{4.5.4}$$

where

$$\Theta_4 := B_{1;2}(u_3 \otimes u_6).$$

The index α in Θ_α associates Θ_α to the α-th interaction operator from the left in (4.5.3). Since we only run the expansion to the n-th level, we have $1 \leq \alpha \leq n$. In this specific case, $n = 4$, the 4th interaction operator is $B_{3;6}$.

Next, $B_{3;5}$ contracts $U_{3,4}^{(1)}\Theta_4$ and $U_{3,4}^{(1)}u_5$,

$$B_{3;5}U_{3,4}^{(5)}((4.5.4)) = (U_{3,4}^{(2)}(u_1 \otimes u_2)) \otimes \Theta_3 \otimes (U_{3,4}^{(1)}u_4), \tag{4.5.5}$$

where

$$\Theta_3 := B_{1;2}((U_{3,4}^{(1)}\Theta_4) \otimes (U_{3,4}^{(1)}u_5)).$$

Then, by the semigroup property, $U_{2,3}^{(i)}U_{3,4}^{(i)} = U_{2,4}^{(i)}$. The operator $B_{2,4}$ contracts $U_{2,4}^{(1)}u_2$ with $U_{2,4}^{(1)}u_4$, which correspond to the 2nd and 5th factors in (4.5.5).

The other factors are left invariant.

$$B_{2;4}U^{(4)}_{2,3}((4.5.5)) = (U^{(1)}_{2,4}u_1) \otimes \Theta_2 \otimes (U^{(1)}_{2,3}\Theta_3), \tag{4.5.6}$$

where

$$\Theta_2 = B_{1,2}(U^{(2)}_{2,4}(u_2 \otimes u_4)).$$

Finally, $B_{1;3}$ contracts $(U^{(1)}_{1,4}u_1)$ and $(U^{(1)}_{1,3}\Theta_3)$ and leaves other factors unchanged.

$$B_{1;3}U^{(3)}_{1,2}((4.5.6)) = \Theta_1 \otimes (U^{(1)}_{1,2}\Theta_2), \tag{4.5.7}$$

where

$$\Theta_1 = B_{1;2}((U^{(1)}_{1,4}u_1) \otimes (U^{(1)}_{1,3}\Theta_3)).$$

Therefore, J^2 can be factorized as

$$J^2 = (U^{(1)}_{0,1}\Theta_1) \otimes (U^{(1)}_{0,2}\Theta_2) := J^1_1 \otimes J^1_2. \tag{4.5.8}$$

In the above expression we may write the factors J^1_j (for $j \leq k = 2$) as one-particle matrices and substitute with $u_i = |\phi\rangle \langle\phi|$, for $i \leq k + n = 6$. Thus, it follows that

$$J^1_1 = U^{(1)}_{0,1}B_{1;2}U^{(2)}_{1,3}B_{2;3}U^{(3)}_{3,4}B_{2;4}(|\phi\rangle \langle\phi|)^{\otimes 4} \tag{4.5.9}$$

where we relabel the index in operators $B_{\sigma_1(r);r}$ such that the interaction operators in (4.5.9) correspond to $B_{1;3}, B_{3;5}, B_{3;6}$ respectively, and most importantly keep the connectivity structure between them. The relabeling function σ_1 (see the notation in (4.4.6)) take values: $\sigma_1(2) = 1, \sigma_1(3) = 2, \sigma_1(4) = 3$. Moreover, for $j = 1$, we perform the relabeling in the same spirit find that

$$J_2^1 = U_{0,2}^{(1)} B_{1;2} U_{2,4}^{(2)} (|\phi\rangle \langle\phi|)^{\otimes 2} \tag{4.5.10}$$

where $\sigma_2(2) = 1$.

We note that for any $l < l'$, the interaction operators $B_{\sigma(l),l}$ and $B_{\sigma(l'),l'}$ in J^2 (associated to the matrix (4.5.2)) belong to the same factor J_j^1 if either $\sigma(l) = \sigma(l')$ or $\sigma(l') = l$. In such cases, we consider them as being *connected*. This connectivity structure is exactly the key point of the Duhamel terms that we want to illustrate using binary tree graphs. Each σ_j can be viewed as the restriction of σ to J_j^1. We call factors that have a free propagator applied to each ϕ (like J_2^1) *regular* and factors that involve the contractions of $(|\phi\rangle \langle\phi|)^{\otimes 2}$ without free propagator in between (like J_1^1) *distinguished*.

4.5.2 Recursive determination of contraction structure

Next, repeating the argument in §4.5.1, we express the kernel of each factor explicitly.

Consider the distinguished factor J_1^1. For $\alpha = 1, 2, 3$, we denote by Θ_α the kernel obtained after contracting a two-particle density matrix to a one-particle matrix via the interaction operator. We will determine Θ_α in the normal form from the last interaction operator

$$\Theta_\alpha(x;x') = \sum_{\beta_\alpha} c^\alpha_{\beta_\alpha} \psi^\alpha_{\beta_\alpha}(x)\overline{\chi^\alpha_{\beta_\alpha}}(x'),\ c^\alpha_{\beta_\alpha} = \pm 1, \tag{4.5.11}$$

where $\psi^\alpha_{\beta_\alpha}$ and $\chi^\alpha_{\beta_\alpha}$ are certain functions that will be recursively determined. First, contracting variables by $B_{2;4}$, we get

$$B_{2;4}(|\phi\rangle\langle\phi|)^{\otimes 4} = (|\phi\rangle\langle\phi|) \otimes \Theta_3 \otimes (|\phi\rangle\langle\phi|) \tag{4.5.12}$$

with

$$\Theta_3(x;x') = |\phi|^2\phi(x)\overline{\phi}(x') - \phi(x)\overline{|\phi|^2\phi}(x') = \sum_{\beta_3=1}^{2} c^3_{\beta_3}\psi^3_{\beta_3}(x)\overline{\chi^3_{\beta_3}}(x').$$

Next, contracting variables by $B_{2;3}$,

$$B_{2;3}U^{(3)}_{3,4}(4.5.12) = (|U_{3,4}\phi\rangle\langle U_{3,4}\phi|) \otimes \Theta_2, \tag{4.5.13}$$

where $U_{i,j} := e^{i(t_i-t_j)\Delta}$ and

$$\begin{aligned}&\Theta_2(x;x')\\ &= \sum_{\beta_3=1}^{2} c^3_{\beta_3}\Big(U_{3,4}\psi^3_{\beta_3}|U_{3,4}\phi|^2\Big)(x)\overline{U_{3,4}\chi^3_{\beta_3}}(x') - c^3_{\beta_3}U_{3,4}\psi^3_{\beta_3}(x)\Big(\overline{U_{3,4}\psi^3_{\beta_3}}|U_{3,4}\chi|^2\Big)(x')\\ &=: \sum_{\beta_2=1}^{4} c^2_{\beta_2}\psi^2_{\beta_2}(x)\overline{\chi^2_{\beta_2}}(x').\end{aligned}$$

Finally, by the first interaction operator $B_{1;2}$,

$$B_{1;2}U^{(2)}_{1,3}(4.5.13) = B_{1,2}\Big(|U_{1,4}\phi\rangle\langle U_{1,4}\phi| \otimes \sum_{\beta_2=1}^{4} c^2_{\beta_2}|U_{1,3}\psi^2_{\beta_2}\rangle\langle U_{1,3}\chi^2_{\beta_2}|\Big) = \Theta_1,$$

where $\Theta_1(x;x')$ is given by

$$\sum_{\beta_2=1}^{4} c^2_{\beta_2}\Big(U_{1,4}\phi U_{1,3}\psi^2_{\beta_2}\overline{U_{1,3}\chi^2_{\beta_2}}\Big)(x)\overline{U_{1,4}\phi}(x') - c^2_{\beta_2}U_{1,4}\phi(x)\Big(\overline{U_{1,4}\phi U_{1,3}\psi^2_{\beta_2}}U_{1,3}\chi^2_{\beta_2}\Big)(x')$$

$$=: \sum_{\beta_1=1}^{8} c^1_{\beta_1} \psi^1_{\beta_1}(x)\overline{\chi^1_{\beta_1}}(x').$$

Therefore, J_1^1 can be represented by

$$J_1^1(x;x') = U^{(1)}_{0,1}\Theta_1(x;x') = \sum_{\beta_1=1}^{8} c^1_{\beta_1} U_{0,1}\psi^1_{\beta_1}(x)\overline{U_{0,1}\chi^1_{\beta_1}}(x'),$$

Similarly, we write the regular factor J_2^1 as

$$J_2^1(\sigma_2;t_2,t_4) = U^{(1)}_{0,1}\tilde{\Theta}_1(x;x') = \sum_{\tilde{\beta}_1=1}^{2} \tilde{c}^1_{\tilde{\beta}_1} U_{0,1}\tilde{\psi}^1_{\tilde{\beta}_1}(x)\overline{U_{0,1}\tilde{\chi}^1_{\tilde{\beta}_1}}(x'),$$

where

$$\begin{aligned}\tilde{\Theta}_1(x;x') &= (|U_{2,4}\phi|^2 U_{2,4}\phi)(x)\overline{U_{2,4}\phi}(x') - U_{2,4}\phi(x)(|U_{2,4}\phi|^2\overline{U_{2,4}\phi})(x') \\ &=: \sum_{\tilde{\beta}_1=1}^{2} \tilde{c}^1_{\tilde{\beta}_1}\tilde{\psi}^1_{\tilde{\beta}_1}(x)\overline{\tilde{\chi}^1_{\tilde{\beta}_1}}(x').\end{aligned}$$

4.5.3 Recursive estimates

Now, we estimate the example (4.5.1) using the structural properties obtained from the previous two subsections. The key tool is the trilinear estimates (Lemma 4.1).

Observe that in the example (4.5.1), the distinguished factor J_1^1 is independent of t_2, and the regular factor J_2^1 depends only on t_2 and t_4 (see (4.5.9) and (4.5.10)). Thus, (4.5.1) can be factored as

$$(4.5.1) = \Big(\int_{[0,T)^2} dt_1 dt_3 \mathrm{Tr}(|S^{(1,-d)}J_1^1|)\Big)\Big(\int_0^T dt_2 \mathrm{Tr}(|S^{(1,-d)}J_2^1|)\Big). \quad (4.5.14)$$

We estimate these two factors separately.

4.5.3.1 Distinguished factor

By §4.5.1 and §4.5.2, we have

$$\int_{[0,T)^2} dt_1 dt_3 \mathrm{Tr}(|S^{(1,-d)} J_1^1|) \leq \sum_{\beta_1=1}^{8} \int_{[0,T)^2} dt_1 dt_3 \|\psi_{\beta_1}^1\|_{H^{-d}} \|\chi_{\beta_1}^1\|_{H^{-d}}, \quad (4.5.15)$$

where for each β_α, only one out of two terms $\psi_{\beta_\alpha}^\alpha$ and $\chi_{\beta_\alpha}^\alpha$ is cubic. Among the eight integrals on the right hand side of (4.5.15), we estimate the following two cases, since all others are similar.

Case 1. Consider the integral whose $\psi_{\beta_\alpha}^\alpha$'s are all cubic, precisely

$$\begin{aligned} \psi_{\beta_1}^1 &= U_{1,4}\phi U_{1,3}\psi_{\beta_2}^2 \overline{U_{1,3}\chi_{\beta_2}^2}, & \chi_{\beta_1}^1 &= U_{1,4}\phi, \\ \psi_{\beta_2}^2 &= U_{3,4}\psi_{\beta_3}^3 |U_{3,4}\phi|^2, & \chi_{\beta_2}^2 &= U_{3,4}\chi_{\beta_3}^3, \\ \psi_{\beta_3}^3 &= |\phi|^2\phi, & \chi_{\beta_3}^3 &= \phi. \end{aligned} \quad (4.5.16)$$

We apply the trilinear estimates (4.3.2) recursively while keeping the $W^{-s_c+\frac{\epsilon}{2},r_\epsilon}$ norm on $\psi_{\beta_\alpha}^\alpha$. Then, we obtain that

$$\begin{aligned} &\int_{[0,T)^2} dt_1 dt_3 \|\psi_{\beta_1}^1\|_{H^{-d}} \|\chi_{\beta_1}^1\|_{H^{-d}} \\ &\lesssim \int_{[0,T)^2} dt_1 dt_3 \|\psi_{\beta_1}^1\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\chi_{\beta_1}^1\|_{H^{s_\epsilon}} \quad \text{(by Sobolev ineq)} \\ &= \int_{[0,T)^2} dt_1 dt_3 \|U_{1,4}\phi U_{1,3}\psi_{\beta_2}^2 \overline{U_{1,3}\chi_{\beta_2}^2}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\phi\|_{H^{s_\epsilon}} \\ &\leq C_0 T^\epsilon \int_0^T dt_3 \|\psi_{\beta_2}^2\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\chi_{\beta_2}^2\|_{H^{s_\epsilon}} \|\phi\|_{H^{s_\epsilon}}^2 \quad \text{(by (4.3.2))} \\ &= C_0 T^\epsilon \int_0^T dt_3 \|U_{3,4}\psi_{\beta_3}^3 |U_{3,4}\phi|^2\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\phi\|_{H^{s_\epsilon}}^3 \\ &\leq (C_0 T^\epsilon)^2 \|\psi_{\beta_3}^3\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\phi\|_{H^{s_\epsilon}}^5 \quad \text{(by (4.3.2))} \end{aligned}$$

$$= (C_0 T^\epsilon)^2 \| |\phi|^2 \phi \|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\phi\|^5_{H^{s_\epsilon}}$$
$$\lesssim (C_0 T^\epsilon)^2 \|\phi\|^8_{H^{s_\epsilon}} \quad \text{(by Sobolev ineq)}.$$

Case 2. Consider the integral whose $\psi^\alpha_{\beta_\alpha}$'s are all linear except the last one, that is,

$$\begin{aligned} \psi^1_{\beta_1} &= U_{1,3}\psi^2_{\beta_2}, \quad \chi^1_{\beta_1} = U_{1,3}\chi^2_{\beta_2}|U_{1,4}\phi|^2, \\ \psi^2_{\beta_2} &= U_{3,4}\psi^3_{\beta_3}, \quad \chi^2_{\beta_2} = U_{3,4}\chi^3_{\beta_3}|U_{3,4}\phi|^2, \\ \psi^3_{\beta_3} &= |\phi|^2\phi, \qquad \chi^3_{\beta_3} = \phi. \end{aligned} \tag{4.5.17}$$

In this case, we first combine linear propagators acting on $\psi^3_{\beta_3}$ so that

$$\psi^1_{\beta_1} = U_{1,3}U_{3,4}(|\phi|^2\phi) = U_{1,4}(|\phi|^2\phi).$$

Then, applying the trilinear estimate (4.3.3) twice, we obtain

$$\begin{aligned} &\int_{[0,T)^2} dt_1 dt_3 \|\psi^1_{\beta_1}\|_{H^{-d}} \|\chi^1_{\beta_1}\|_{H^{-d}} \\ &\lesssim \int_{[0,T)^2} dt_1 dt_3 \|U_{1,4}(|\phi|^2\phi)\|_{H^{-d}} \|U_{1,3}\chi^2_{\beta_2}|U_{1,4}\phi|^2\|_{H^{s_\epsilon}} \\ &= \int_{[0,T)^2} dt_1 dt_3 \||\phi|^2\phi\|_{H^{-d}} \|U_{1,3}\chi^2_{\beta_2}|U_{1,4}\phi|^2\|_{H^{s_\epsilon}} \\ &\le C_0 T^\epsilon \int_0^T dt_3 \||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\chi^2_{\beta_2}\|_{H^{s_\epsilon}} \|\phi\|^2_{H^{s_\epsilon}} \quad \text{(by (4.3.3))} \\ &\le (C_0 T^\epsilon)^2 \||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\phi\|^5_{H^{s_\epsilon}} \quad \text{(by (4.3.3))} \\ &\lesssim (C_0 T^\epsilon)^2 \|\phi\|^8_{H^{s_\epsilon}} \quad \text{(by Sobolev ineq)}, \end{aligned}$$

which is the same bound as in Case 1.

Similarly, one can show that the other six integrals satisfy the same bound. Then, it follows that

$$\int_{[0,T)^2} dt_1 dt_3 \mathrm{Tr}(|S^{(1,-d)} J_1^1|) \lesssim 8(C_0 T^\epsilon)^2 \|\phi\|_{H^{s_\epsilon}}^8.$$

4.5.3.2 Regular factor

For the regular factor, we have

$$\int_0^T dt_2 \mathrm{Tr}(|S^{(1,-d)} J_2^1|) \leq \sum_{\tilde{\beta}_1=1}^{2} \int_0^T dt_2 \|\tilde{\psi}_{\tilde{\beta}_1}^1\|_{H^{-d}} \|\tilde{\chi}_{\tilde{\beta}_1}^1\|_{H^{-d}}, \tag{4.5.18}$$

where for each $\tilde{\beta}_1$, only one out of two terms $\tilde{\psi}_{\tilde{\beta}_1}^1$ and $\tilde{\chi}_{\tilde{\beta}_1}^1$ is cubic. For instance, when $\tilde{\psi}_{\tilde{\beta}_1}^1 = |U_{2,4}\phi|^2 U_{2,4}\phi$ and $\tilde{\chi}_{\tilde{\beta}_1}^1 = U_{2,4}\phi$, it follows from the trilinear estimate (4.3.3) that

$$\int_0^T dt_2 \|\tilde{\psi}_{\tilde{\beta}_1}^1\|_{H^{-d}} \|\tilde{\chi}_{\tilde{\beta}_1}^1\|_{H^{-d}} \leq \int_0^T dt_2 \||U_{2,4}\phi|^2 U_{2,4}\phi\|_{H^{s_\epsilon}} \|U_{2,4}\phi\|_{H^{s_\epsilon}} \leq C_0 T^\epsilon \|\phi\|_{H^{s_\epsilon}}^4.$$

Similarly, one can also show that the other integral satisfies the same bound. Therefore, we get

$$\int_0^T dt_2 \mathrm{Tr}(|S^{(1,-d)} J_2^1|) \leq 2C_0 T^\epsilon \|\phi\|_{H^{s_\epsilon}}^4$$

4.5.3.3 Conclusion

Going back to (4.5.14), we conclude that

$$(4.5.1) \lesssim 2^4 \cdot (C_0 T^\epsilon)^3 \|\phi\|_{H^{s_\epsilon}}^{12}.$$

4.6 Binary tree graphs for the general case

In order to prove Lemma 4.5 in the general case, we proceed as in [5], and use binary tree graphs. These graphs will help us keep track of the contraction operations applied iteratively in the Duhamel expansion (4.4.5).

4.6.1 The binary tree graphs

We begin by recalling that, by (4.4.6), J^k is given by

$$J^k(\underline{t}_n;\sigma) = U_{0,1}^{(k)} B_{\sigma(k+1);k+1} U_{1,2}^{(k+1)} B_{\sigma(k+2);k+2} \cdots U_{n-1,n}^{(k+n-1)} B_{\sigma(k+n);k+n} (|\phi\rangle \langle\phi|)^{\otimes(k+n)},$$

where

$$(|\phi\rangle\langle\phi|)^{\otimes(k+n)}(\mathbf{x}_{k+n};\mathbf{x}'_{k+n}) = \prod_{i=1}^{k+n} (|\phi\rangle\langle\phi|)(x_i;x'_i)$$

is a product of one-particle kernels. Since the free propagator U and the contraction operators B preserve the product structure, it follows that we can also decompose

$$J^k(t,t_1,\ldots,t_r;\sigma;\mathbf{x}_k;\mathbf{x}'_k) = \prod_{j=1}^{k} J_j^1(t,t_{\ell_{j,1}},\ldots,t_{l_{j,m_j}};\sigma_j;x_j;x'_j) \tag{4.6.1}$$

into a product of one-particle kernels J_j^1. We associate to this decomposition k disjoint binary tree graphs $\tau_1,\tau_2,\ldots,\tau_k$. These graphs appear as *skeleton graphs* in [14, 15, 16, 17]. As in [5], we assign *root, internal,* and *leaf* vertices to each tree τ_j.

- A *root* vertex labeled as W_j, $j=1,2,\cdots,k$, to represent $J_j^1(x_j,x'_j)$.
- An *internal* vertex labeled by v_l, $l=1,2,\cdots,n$, corresponding to $B_{\sigma(k+l),k+l}$ and attached to the time variable t_l.
- A *leaf* vertex u_i, $i=1,2,\cdots,k+n$, representing each factor $(|\phi\rangle\langle\phi|)(x_i;x'_i)$.

Next, we connect the vertices with *edges*, as described below.

- If v_l is the smallest value of l such that $\sigma(k+l) = j$, then we connect v_l to the root vertex W_j and write $W_j \sim v_l$ (or equivalently $W_j \sim B_{\sigma(k+l);k+l}$). If there is no internal vertex connected to a root vertex W_j, then we connect W_j to the leaf u_j, and write $W_j \sim u_j$.

- For any $1 < l \leq n$, if $\exists l' > l$ such that $\sigma(k+l) = \sigma(k+l')$ or $\sigma(k+l') = k+l$, then we connect v_l and $v_{l'}$ and write $v_l \sim v_{l'}$ (or equivalently $B_{\sigma(k+l);k+l} \sim B_{\sigma(k+l');k+l'}$). In this case, we call v_l the *parent vertex* of $v_{l'}$, and $v_{l'}$ the *child vertex* of v_l. We denote the two child vertices of v_l by $v_{k_-(l)}$ and $v_{k_+(l)}$, with $k_-(l) < k_+(l)$.

- When there is no internal vertex with $l' > l$ and $k + l = \sigma(k+l')$, we connect v_l to the leaf vertex u_{k+l} and write $v_l \sim u_{k+l}$ (or equivalently $B_{\sigma(k+l);k+l} \sim u_{k+l}$). If there is no internal vertex with $l' > l$ and $\sigma(k+l) = \sigma(k+l')$, then we connect v_l to the leaf vertex $u_{\sigma(k+l)}$ and write $v_l \sim u_{\sigma(k+l)}$ (or equivalently $B_{\sigma(k+l);k+l} \sim u_{\sigma(k+l)}$).

It follows from the construction above that each root vertex has only one child vertex, and each internal vertex has exactly two child vertices (which can be internal and leaf). We call the tree τ_j *distinguished* if $v_n \in \tau_j$, and *regular* if $v_n \notin \tau_j$. The two leaves connected to v_n are called *distinguished leaf vertices*, and all other leaves are called *regular leaf vertices*. Clearly, there are $k-1$ regular trees and one distinguished tree in each binary tree graph.

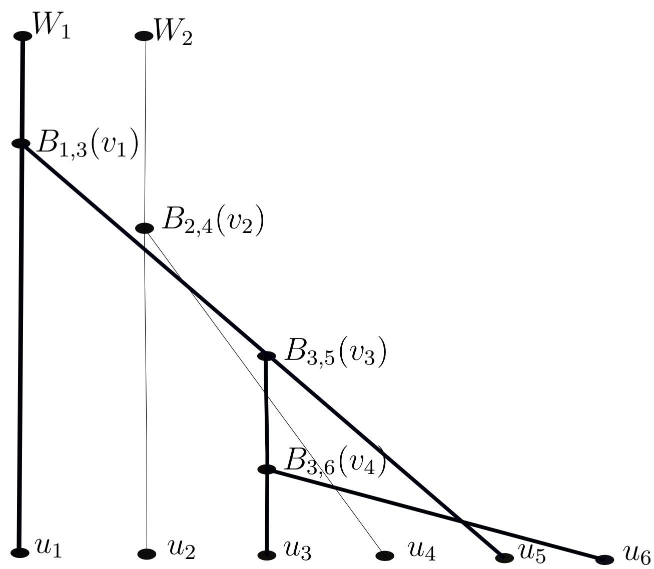

Figure 4.1: An example binary tree graphs of J^k. It is a disjoint union of two trees τ_1 and τ_2 with root vertices W_1 and W_2, respectively. Each tree corresponds to a one-particle kernel in the example in §4.5, where $k = 2$ and $n = 4$.

A sample binary tree graph is given in Figure 4.1, for J^k as in (4.5.3). Each tree τ_j has root vertex W_j, for $j = 1, 2$. The two leaf vertices u_3 and u_6 and the internal vertex v_4 (or $B_{3;6}$) are distinguished. τ_1 is the distinguished tree, and is drawn with thick edges.

4.6.2 The distinguished one particle kernel J_j^1

Let τ_j denote the distinguished tree graph. It has m_j internal vertices $(v_{\ell_j,\alpha})_{\alpha=1}^{m_j}$ and m_j+1 leaf vertices $(u_{j,i})_{i=1}^{m_j+1}$. We enumerate the internal vertices with $\alpha \in \{1, \ldots, m_j\}$ and the leaf vertices with $\alpha \in \{m_j+1, \ldots, 2m_j+2\}$. To simplify notation, we refer to the vertex $v_{j,\alpha}$ by its label α. We observe that

J_j^1 has the form

$$\begin{aligned}
&J_j^1(t, t_{\ell_{j,1}}, \ldots, t_{\ell_{j,m_j}}; \sigma_j) &(4.6.2)\\
&= U^{(1)}(t - t_{\ell_{j,1}}) \cdots U^{(1)}(t_{\ell_{j,1}-1} - t_{\ell_{j,1}}) B_{\sigma_j(2);2} \cdots \\
&\qquad \cdots B_{\sigma_j(\alpha);\alpha} U^{(\alpha)}(t_{\ell_{j,\alpha-1}} - t_{\ell_{j,\alpha-1}+1}) \cdots U^{(\alpha)}(t_{\ell_{j,\alpha}-1} - t_{\ell_{j,\alpha}}) B_{\sigma_j(\alpha+1);\alpha+1} \cdots \\
&\qquad \cdots U^{(m_j)}(t_{\ell_{j,m_j}-1} - t_{l_{j,m_j}}) B_{\sigma_j(m_j+1);m_j+1} (|\phi\rangle\langle\phi|)^{\otimes(m_j+1)}.
\end{aligned}$$

By the group property

$$U^{(\alpha)}(t)U^{(\alpha)}(s) = U^{(\alpha)}(t+s),$$

and the fact that $\sigma_j(2) = 1$, (4.6.2) reduces to

$$\begin{aligned}
&J_j^1(t, t_{\ell_{j,1}}, \ldots, t_{\ell_{j,m_j}}; \sigma_j) &(4.6.3)\\
&= U^{(1)}(t - t_{\ell_{j,1}}) B_{1;2} \cdots \\
&\qquad \cdots B_{\sigma_j(\alpha);\alpha} U^{(\alpha)}(t_{\ell_{j,\alpha-1}} - t_{\ell_{j,\alpha}}) B_{\sigma_j(\alpha+1);\alpha+1} \cdots \\
&\qquad \cdots U^{(m_j)}(t_{\ell_{j,m_j}-1} - t_{l_{j,m_j}}) B_{\sigma_j(m_j+1);m_j+1} (|\phi\rangle\langle\phi|)^{\otimes(m_j+1)},
\end{aligned}$$

where $\ell_{j,m_j} = r$.

4.6.3 Definition of the kernels Θ_α at the vertices of the distinguished tree graph

In this section, we proceed as in [5], and recursively assign a kernel Θ_α to each vertex α of the distinguished tree graph. The kernels at the vertices of the regular tree graph are defined similarly. We begin by assigning the kernel

$$\Theta_\alpha(x; x') := \phi(x)\overline{\phi}(x')$$

to the leave vertex with label $\alpha \in \{m_j+1, \ldots, 2m+j+2\}$ (corresponding to $u_{j,\alpha-m_j}$).

Next, we determine Θ_{m_j} at the distinguished vertex $\alpha = m_j$ from the term on the last line of (4.6.3), given by

$$
\begin{aligned}
B_{\sigma_j(m_j+1);m_j+1}(|\phi\rangle\langle\phi|)^{\otimes(m_j+1)} = (|\phi\rangle\langle\phi|)^{\otimes(\sigma_j(m_j+1)-1)} \otimes \Theta_{m_j} \\
\otimes (|\phi\rangle\langle\phi|)^{\otimes(m_j+1-\sigma_j(m_j+1)-1)}
\end{aligned}
$$

where

$$
\Theta_{m_j}(x;x') := \tilde{\psi}(x)\overline{\phi}(x') - \phi(x)\overline{\tilde{\psi}}(x') \tag{4.6.4}
$$

with $\tilde{\psi} := |\phi|^2\phi$. It is obtained from contracting two copies of $|\phi\rangle\langle\phi|$ at the two leaf vertices $\kappa_-(m_j), \kappa_+(m_j)$ which have m_j as their parent vertex.

Now we are ready to begin the induction. Let $\alpha \in \{1, \ldots, m_j - 1\}$. Suppose that the kernels $\Theta_{\alpha'}$ have been determined for all $\alpha' > \alpha$. We let $\kappa_-(\alpha), \kappa_+(\alpha)$ label the two child vertices (of internal or leaf type) of α,

$$
\sigma_j(\alpha) = \sigma_j(\kappa_-(\alpha)) \quad , \quad \alpha = \sigma_j(\kappa_+(\alpha)).
$$

Since $\Theta_{\kappa_-(\alpha)}$ and $\Theta_{\kappa_+(\alpha)}$ have already been determined, we can now define

$$
\begin{aligned}
&\Theta_\alpha(x;x') \\
&= B_{1;2}((U^{(1)}(t_\alpha - t_{\kappa_-(\alpha)}) \otimes (U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)}\Theta_{\kappa_+(\alpha)}))(x;x') \\
&= (U^{(1)}(t_\alpha - t_{\kappa_-(\alpha)})\Theta_{k_-(\alpha)})(x;x')[(U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)})\Theta_{\kappa_+(\alpha)})(x;x) \\
&\qquad\qquad - (U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)})\Theta_{\kappa_+(\alpha)})(x';x')].
\end{aligned}
$$

The induction ends when we obtain the kernel Θ_1 at $\alpha = 1$.

4.6.4 Key properties of the kernels Θ_α

As in [5], we observe that the kernels Θ_α satisfy the following properties.

- Θ_α can be written as a sum of differences of factorized kernels

$$\Theta_\alpha(x;x') = \sum_{\beta_\alpha} c^\alpha_{\beta_\alpha} \chi^\alpha_{\beta_\alpha}(x) \overline{\psi^\alpha_{\beta_\alpha}}(x') \tag{4.6.5}$$

with at most $2^{m_j-\alpha}$ nonzero coefficients $c^\alpha_{\beta_\alpha} \in \{1,-1\}$.

- The product $\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}}(x')$ in (4.6.5) above is either of the form

$$\begin{aligned}\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}}(x') = (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x)\overline{(U_{\alpha;\kappa_-(\alpha)}\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})}(x') \\ (U_{\alpha;\kappa_+(\alpha)}\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})(x)\overline{(U_{\alpha;\kappa_+(\alpha)}\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})}(x)\end{aligned} \tag{4.6.6}$$

or

$$\begin{aligned}\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}}(x') = (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x)\overline{(U_{\alpha;\kappa_-(\alpha)}\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})}(x') \\ (U_{\alpha;\kappa_+(\alpha)}\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})(x')\overline{(U_{\alpha;\kappa_+(\alpha)}\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})}(x')\end{aligned} \tag{4.6.7}$$

for some values of $\beta_{\kappa_-(\alpha)}, \beta_{\kappa_+(\alpha)}$ that depend on β_α. Observe that above, the function $\chi^\alpha_{\beta_\alpha}$ is either of the cubic form

$$\begin{aligned}\chi^\alpha_{\beta_\alpha}(x) = (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x) \\ (U_{\alpha;\kappa_+(\alpha)}\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})(x)\overline{(U_{\alpha;\kappa_+}(\alpha)\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+}})}(x)\end{aligned} \tag{4.6.8}$$

or the linear form

$$\chi^\alpha_{\beta_\alpha}(x) = (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x). \tag{4.6.9}$$

Accordingly, $\psi^\alpha_{\beta_\alpha}$ respectively is either of linear or cubic form, and the product $\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}}(x')$ always has quartic form (4.6.6) or (4.6.7).

- We call the functions $\chi^\alpha_{\beta_\alpha}, \psi^\alpha_{\beta_\alpha}$ in the sum (4.6.5) *distinguished* if they are a function of $|\phi|^2\phi$. In the product on the right hand side of (4.6.6), respectively (4.6.7), at most one of the four pieces contains a distinguished function. Indeed, this is true for all regular leaf vertices, and for the distinguished vertex (4.6.4). By induction along decreasing values of α, it is also true for the internal vertices.

4.7 Proof of Lemma 4.5

In this section, we prove Lemma 4.5. We begin by considering the contribution of each factor J^1_j on the right hand side of (4.6.1) separately. One of these factors is distinguished, and will be dealt with in Proposition 4.6 below. Proposition 4.9 will be for the regular factors.

We note that the analog of Proposition 4.6 in [5] has a shorter proof. This is because, where the authors of [5] work in L^2, we work in $W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}$ to achieve lower regularity. Under the norm $W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}$, the linear propagators $e^{it\Delta}$ are no longer isometrics, so we need to carefully rearrange them to ensure they do not interfere with our proof. This occurs in case 2 of our proof of Lemma 4.8.

We begin with Proposition 4.6, which addresses the contribution of the distinguished factor J^1_j. We prove Proposition 4.6 by induction. Lemma 4.7 will serve as our first induction step, and Lemma 4.8 will serve as the remainder of our proof by induction.

Proposition 4.6. *Let $d \geq 3$. Then, for the distinguished tree τ_j, we have the bound*

$$\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\left(\left| S^{(1,-d)} J_j^1(t, t_1, \cdots, t_{m_j}; \sigma_j) \right| \right)$$
$$\leq 2^{m_j-1} C^{m_j-1} T^{\epsilon(m_j-1)} \|\phi\|_{H^{s_\epsilon}}^{2m_j-1} \||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}. \quad (4.7.1)$$

Similarly, when $d = 2$, we have the bound

$$\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\left(\left| S^{(1,-d)} J_j^1(t, t_1, \cdots, t_{m_j}; \sigma_j) \right| \right)$$
$$\leq 2^{m_j-1} C^{m_j-1} T^{\frac{1}{3}(m_j-1)} \|\phi\|_{H^{1/3}}^{2m_j-1} \||\phi|^2\phi\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}),r_\epsilon}}, \quad (4.7.2)$$

and, when $d = 1$, we have the bound

$$\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\left(\left| S^{(1,-d)} J_j^1(t, t_1, \cdots, t_{m_j}; \sigma_j) \right| \right)$$
$$\leq 2^{m_j-1} C^{m_j-1} T^{\frac{1}{2}(m_j-1)} \|\phi\|_{L^2}^{2m_j-1} \||\phi|^2\phi\|_{L^1}. \quad (4.7.3)$$

Proof. For $d \geq 3$, Proposition 4.6 follows immediately from Lemma 4.8 below. Indeed, in the statement of Lemma 4.8, there are at most 2^{m_j-1} terms in the sum over β_1.

Observe that in the proofs of Lemmas 4.7 and 4.8, we use the bounds for $d \geq 3$ presented in Lemma 4.1. The proof of Proposition 4.6 for $d = 1, 2$ is analogous (we use the corresponding bounds for $d = 1, 2$ presented in Lemma 4.1). □

We now prove Lemma 4.7, which will serve as the first induction step in our proof of Lemma 4.6.

Lemma 4.7. *Let $d \geq 3$. Then, the distinguished factor*

$$J_j^1(\underline{t}_n;\sigma_j;x,x') = U^{(1)}(t-t_1)\sum_{\beta_1} c_{\beta_1}^1 \psi_{\beta_1}^1(x)\chi_{\beta_1}^1(x')$$

satisfies the following. For each value of β_1, either there exits a non-negative integer $\ell < m_j - 1$ such that

$$\begin{aligned}
&\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\Big(\Big| S^{(1,-d)} U^{(1)}(t-t_1) c_{\beta_1}^1 |\psi_{\beta_1}^1\rangle\langle\chi_{\beta_1}^1| \Big| \Big) \\
&\leq (CT^\epsilon)^\ell \sum_{\beta_1} \int_{[0,T)^{m_j-\ell-1}} \|(U_{\ell+2} f_{\ell+2}^1)(U_{\ell+2} f_{\ell+2}^2)(U_{\ell+2} f_{\ell+2}^3)\|_{W^{-s_c+\frac{\epsilon}{2},r_\epsilon}} \\
&\qquad \times \|U_{\ell+2} f_{\ell+2}^2\|_{H^{s_\epsilon}} \cdots \|U_{\ell+2} f_{\ell+2}^{2\ell+4}\|_{H^{s_\epsilon}} dt_{\ell+1} \cdots dt_{m_j-1}, \qquad (4.7.4)
\end{aligned}$$

where the functions f are defined in terms of the functions $\psi_{\beta_\alpha}^\alpha$ and $\chi_{\beta_\alpha}^\alpha$ as described in the proof below, or

$$\begin{aligned}
&\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\Big(\Big| S^{(1,-d)} U^{(1)}(t-t_1) c_{\beta_1}^1 |\psi_{\beta_1}^1\rangle\langle\chi_{\beta_1}^1| \Big| \Big) \\
&\leq C^{m_j-1} T^{\epsilon(m_j-1)} \|\phi\|_{H^{s_\epsilon}}^{2m_j-1} \||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}. \qquad (4.7.5)
\end{aligned}$$

Moreover, $f_{\ell+2}^1$ is the only distinguished function on the right hand side of (4.7.4).

Proof. We recall that $U_{i,j} := e^{i(t_i-t_j)\Delta}$, and let $U_j := U_{j,j+1}$. We have

$$\begin{aligned}
&\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\Big(\Big| S^{(1,-d)} U^{(1)}(t-t_1) c_{\beta_1}^1 |\psi_{\beta_1}^1\rangle\langle\chi_{\beta_1}^1| \Big| \Big) \\
&\leq \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|\psi_{\beta_1}^1\|_{H^{-d}} \|\chi_{\beta_1}^1\|_{H^{-d}}. \qquad (4.7.6)
\end{aligned}$$

Now, we recall from §4.6.4 that one of functions $\psi_{\beta_1}^1, \chi_{\beta_1}^1$ is distinguished. Moreover the distinguished function is either of the cubic form (4.6.8) or of the linear form (4.6.9). We will now label the distinguished function f_1^1 and the regular function f_1^2.

Case 1: f_1^1 is cubic. If f_1^1 is cubic, then, by (4.6.6) and (4.6.7), f_1^1 and f_1^2 are of the form

$$\begin{aligned} f_1^1 &= (U_2 f_2^1)(U_2 f_2^2)(U_2 f_2^3), \\ f_1^2 &= U_2 f_2^4. \end{aligned}$$

As in §4.5, we apply the $W^{-s_c+\frac{\epsilon}{2}, r_\epsilon}$ norm to the distinguished function f_1^1 and the H^{s_ϵ} norm to the regular function f_1^2 and find that

$$\begin{aligned} (4.7.6) &= \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|f_1^1\|_{H^{-d}} \|f_1^2\|_{H^{-d}} \\ &= \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|(U_2 f_2^1)(U_2 f_2^2)(U_2 f_2^3)\|_{H^{-d}} \|U_2 f_2^4\|_{H^{-d}} \\ &\leq C \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|(U_2 f_2^1)(U_2 f_2^2)(U_2 f_2^3)\|_{W^{-(s_c+\frac{\epsilon}{2}), r_\epsilon}} \|U_2 f_2^4\|_{H^{s_\epsilon}}, \end{aligned}$$

which is of the form (4.7.4).

Case 2: f_1^2 is cubic. In this case, we have that f_1^1 and f_1^2 are of the form

$$\begin{aligned} f_1^1 &= U_2 f_2^1, \\ f_1^2 &= (U_2 f_2^2)(U_2 f_2^3)(U_2 f_2^4). \end{aligned}$$

Since f_1^1 is distinguished, there exists $\ell \geq 1$ such that

$$f_2^1 = U_3 f_3^1, \;\; f_3^1 = U_4 f_4^1, ..., f_\ell^1 = U_{\ell+1} f_{\ell+1}^1,$$

and

$$f^1_{\ell+1} = (U_{\ell+2} f^1_{\ell+2})(U_{\ell+2} f^2_{\ell+2})(U_{\ell+2} f^3_{\ell+2}) \text{ or } f^1_{\ell+1} = |\phi|^2\phi, \tag{4.7.7}$$

where $f^1_{\ell+2}$ (or $f^2_{\ell+2}$ or $f^3_{\ell+2}$) is a distinguished function. Thus, combining all propagators acting on $f^1_{\ell+1}$, we write

$$f^1_1 = U_{1,\ell+2} f^1_{\ell+1}.$$

Again, we apply the $W^{-s_c+\frac{\epsilon}{2},r_\epsilon}$ norm to the distinguished function f^1_1 and the H^{s_ϵ} norm to the regular function f^2_1 and find that

$$\begin{aligned}
(4.7.6) &= \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|f^1_1\|_{H^{-d}} \|f^2_1\|_{H^{-d}} \\
&= \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|f^1_{\ell+1}\|_{H^{-d}} \|(U_2 f^2_2)(U_2 f^3_2)(U_2 f^4_2)\|_{H^{-d}} \\
&\lesssim \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|f^1_{\ell+1}\|_{W^{-s_c+\frac{\epsilon}{2},r_\epsilon}} \|(U_2 f^2_2)(U_2 f^3_2)(U_2 f^4_2)\|_{H^{s_\epsilon}}.
\end{aligned} \tag{4.7.8}$$

Since $f_{\ell+1}$ doesn't depend on $t_1, \ldots, t_\ell$, we find that after ℓ applications of (4.3.3),

$$(4.7.8) \leq (CT^\epsilon)^\ell \int_{[0,T)^{m_j-\ell-1}} dt_{\ell+1} \cdots dt_{m_j-1} \|f^1_{\ell+1}\|_{W^{-s_c+\frac{\epsilon}{2},r_\epsilon}} \|f^2_{\ell+1}\|_{H^{s_\epsilon}} \cdots \|f^{2\ell+4}_{\ell+1}\|_{H^{s_\epsilon}}. \tag{4.7.9}$$

If $f^1_{\ell+1} = |\phi|^2\phi$, then it follows from the binary tree graph structure presented in §4.6 that $\ell = m_j - 1$ and $f^{\ell''}_{\ell+1} = \phi$ for $\ell'' \geq 2$, and so we have proven (4.7.5). Otherwise, if $f^1_{\ell+1} = (U_{\ell+2} f^1_{\ell+2})(U_{\ell+2} f^2_{\ell+2})(U_{\ell+2} f^3_{\ell+2})$, then we have that

$$\begin{aligned}
&(4.7.9) \\
&\leq (CT^\epsilon)^\ell \int_{[0,T)^{m_j-\ell-1}} \|(U_{\ell+2} f^1_{\ell+2})(U_{\ell+2} f^2_{\ell+2})(U_{\ell+2} f^3_{\ell+2})\|_{W^{-s_c+\frac{\epsilon}{2},r_\epsilon}}
\end{aligned}$$

$$
\begin{aligned}
&\times \|f_{\ell+1}^2\|_{H^{s_\epsilon}} \cdots \|f_{\ell+1}^{2\ell+4}\|_{H^{s_\epsilon}} dt_{\ell+1} \cdots dt_{m_j-1} \\
= (CT^\epsilon)^\ell &\int_{[0,T)^{m_j-\ell-1}} \|(U_{\ell+2} f_{\ell+2}^1)(U_{\ell+2} f_{\ell+2}^2)(U_{\ell+2} f_{\ell+2}^3)\|_{W^{-s_c+\frac{\epsilon}{2}, r_\epsilon}} \\
&\times \|U_{\ell+2} f_{\ell+2}^2\|_{H^{s_\epsilon}} \cdots \|U_{\ell+2} f_{\ell+2}^{2\ell+4}\|_{H^{s_\epsilon}} dt_{\ell+1} \cdots dt_{m_j-1},
\end{aligned}
$$

which is of the form (4.7.4). □

In Lemma 4.8, we complete the induction process. Observe that in the proof below, we proceed as in the proof of Lemma 4.7. In each induction step, we apply the $W^{s_c+\frac{\epsilon}{2}, r_\epsilon}$ norm to the distinguished function, and the H^{s_ϵ} norm to the regular functions.

Lemma 4.8. *Let $d \geq 3$. Then, the distinguished factor*

$$
J_j^1(\underline{t}_n; \sigma_j; x, x') = U^{(1)}(t - t_1) \sum_{\beta_1} c_{\beta_1}^1 \psi_{\beta_1}^1(x) \chi_{\beta_1}^1(x')
$$

satisfies the following. For each value of β_1,

$$
\begin{aligned}
&\int_{[0,T)^{m_j-1}} dt_1 \ldots dt_{m_j-1} \mathrm{Tr}\left(\left| S^{(1,-d)} U^{(1)}(t - t_1) c_{\beta_1}^1 |\psi_{\beta_1}^1\rangle\langle\chi_{\beta_1}^1| \right| \right) \\
&\leq C^{m_j-1} T^{\epsilon(m_j-1)} \|\phi\|_{H^{s_\epsilon}}^{2m_j-1} \||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}), r_\epsilon}}.
\end{aligned}
\tag{4.7.10}
$$

Proof. By Lemma 4.7, we have that for each β_1, either (4.7.10) holds, or there is a non-negative integer $\ell < m_j - 1$ such that

$$
\begin{aligned}
&\int_{[0,T)^{m_j-1}} dt_1 \ldots dt_{m_j-1} \mathrm{Tr}\left(\left| S^{(1,-d)} U^{(1)}(t - t_1) c_{\beta_1}^1 |\psi_{\beta_1}^1\rangle\langle\chi_{\beta_1}^1| \right| \right) \\
&\leq (CT^\epsilon)^\ell 2^{m_j-1} \int_{[0,T)^{m_j-\ell-1}} \|(U_{\ell+2} f_{\ell+2}^1)(U_{\ell+2} f_{\ell+2}^2)(U_{\ell+2} f_{\ell+2}^3)\|_{W^{-s_c+\frac{\epsilon}{2}, r_\epsilon}} \\
&\qquad \times \|U_{\ell+2} f_{\ell+2}^2\|_{H^{s_\epsilon}} \cdots \|U_{\ell+2} f_{\ell+2}^{2\ell+4}\|_{H^{s_\epsilon}} dt_{\ell+1} \cdots dt_{m_j-1},
\end{aligned}
\tag{4.7.11}
$$

where $f^1_{\ell+2}$ is the only distinguished function on the right hand side of (4.7.11). We recall from §4.6 that $f^1_{\ell+2}$ is either of the cubic form (4.6.8) or the linear for (4.6.9).

Now, we will proceed by induction, and show that in each induction step, we can bound 4.7.11 by an expression of the same form, but with a larger value of ℓ. In the last induction step, we find that (4.7.15) holds, which completes the proof of (4.7.10). Indeed, this follows from the binary tree graph structure presented in §4.6.

Case 1: $f^1_{\ell+2}$ is cubic. If $f^1_{\ell+2}$ is cubic, then

$$\begin{aligned}
&f^1_{\ell+2} = (U_{\ell+3}f^1_{\ell+3})(U_{\ell+3}f^2_{\ell+3})(U_{\ell+3}f^3_{\ell+3}),\\
&f^2_{\ell+2} = U_{\ell+3}f^4_{\ell+3},\ f^3_{\ell+2} = U_{\ell+3}f^5_{\ell+3}, ...,\ f^{2\ell+4}_{\ell+2} = U_{\ell+3}f^{2\ell+6}_{\ell+3}.
\end{aligned}$$

Since $f^1_{\ell+2}$ is distinguished, one of $f^1_{\ell+3}, f^2_{\ell+3}, f^3_{\ell+3}$ is distinguished, say $f^1_{\ell+3}$. Then, applying (4.3.2), we get the integral of the form (4.7.11) back:

$$\begin{aligned}
&(4.7.11)\\
&\lesssim (CT^\epsilon)^{\ell+1}2^{m_j-1}\int_{[0,T)^{m_j-\ell-2}} dt_{\ell+2}\cdots dt_{m_j-1}\\
&\qquad\qquad\times \|f^1_{\ell+2}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}\|f^2_{\ell+2}\|_{H^{s_\epsilon}}\cdots\|f^{2\ell+4}_{\ell+2}\|_{H^{s_\epsilon}}\\
&= (CT^\epsilon)^{\ell+1}2^{m_j-1}\int_{[0,T)^{m_j-\ell-2}} dt_{\ell+2}\cdots dt_{m_j-1}\\
&\quad\times\|(U_{\ell+3}f^1_{\ell+3})(U_{\ell+3}f^2_{\ell+3})(U_{\ell+3}f^3_{\ell+3})\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}\|f^4_{\ell+3}\|_{H^{s_\epsilon}}\cdots\|f^{2\ell+6}_{\ell+3}\|_{H^{s_\epsilon}}.
\end{aligned}$$

Case 2: $f^2_{\ell+2}$ is cubic. If $f^1_{\ell+2}$ is cubic, then

$$f^1_{\ell+2} = U_{\ell+3}f^1_{\ell+3},$$

$$f^2_{\ell+2} = (U_{\ell+3} f^2_{\ell+3})(U_{\ell+3} f^3_{\ell+3})(U_{\ell+3} f^4_{\ell+3}),$$
$$f^3_{\ell+2} = U_{\ell+3} f^5_{\ell+3}, \ldots,\; f^{2\ell+4}_{\ell+2} = U_{\ell+3} f^{2\ell+6}_{\ell+3}.$$

Since $f^1_{\ell+2}$ is distinguished, there exists $\ell' \geq 1$ such that

$$f^1_{\ell+3} = U_{\ell+4} f^1_{\ell+4}, f^1_{\ell+4} = U_{\ell+5} f^1_{\ell+5}, \ldots, f^1_{\ell+1+\ell'} = U_{\ell+2+\ell'} f^1_{\ell+2+\ell'},$$

and

$$f^1_{\ell+2+\ell'} = (U_{\ell+3+\ell'} f^1_{\ell+3+\ell'})(U_{\ell+3+\ell'} f^2_{\ell+3+\ell'})(U_{\ell+3+\ell'} f^3_{\ell+3+\ell'}) \text{ or } f^1_{\ell+2+\ell'} = |\phi|^2\phi, \tag{4.7.12}$$

where $f^1_{\ell+3+\ell'}$ is a distinguished function. Thus, combining all linear propagators acting on $f^1_{\ell+2+\ell'}$, we write

$$f^1_{\ell+2} = U_{\ell+2,\ell+3+\ell'} f^1_{\ell+2+\ell'}.$$

Then, applying (4.3.2) and (4.3.3), we obtain

$$\begin{aligned}
&(4.7.11)\\
&\leq (CT^\epsilon)^{\ell+1} 2^{m_j-1} \int_{[0,T)^{m_j-\ell-2}} dt_{\ell+2} \cdots dt_{m_j-1}\\
&\qquad \times \|f^1_{\ell+2+\ell'}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|f^2_{\ell+2}\|_{H^{s_\epsilon}} \cdots \|f^{2\ell+4}_{\ell+2}\|_{H^{s_\epsilon}}\\
&\leq (CT^\epsilon)^{\ell+2} 2^{m_j-1} \int_{[0,T)^{m_j-\ell-3}} dt_{\ell+3} \cdots dt_{m_j-1}\\
&\qquad \times \|f^1_{\ell+2+\ell'}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|f^2_{\ell+3}\|_{H^{s_\epsilon}} \cdots \|f^{2\ell+6}_{\ell+3}\|_{H^{s_\epsilon}},
\end{aligned} \tag{4.7.13}$$

where, in the second inequality, we applied (4.3.3) to the cubic regular function $f^2_{\ell+2}$. After $\ell' - 1$ applications of (4.3.3), we find that

$$(4.7.13) \leq (CT^\epsilon)^{\ell+1+\ell'} 2^{m_j-1} \int_{[0,T)^{m_j-\ell-2-\ell'}} dt_{\ell+2+\ell'} \cdots dt_{m_j-1}$$

$$\times \|f^1_{\ell+2+\ell'}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|f^2_{\ell+2+\ell'}\|_{H^{s_\epsilon}} \cdots \|f^{2\ell+2\ell'+4}_{\ell+2+\ell'}\|_{H^{s_\epsilon}}. \tag{4.7.14}$$

If

$$f^1_{\ell+2+\ell'} = |\phi|^2\phi, \tag{4.7.15}$$

then it follows from the binary tree graph structure presented in §4.6 that $\ell+2+\ell' = m_j$ and $f^{\ell''}_{\ell+2+\ell'} = \phi$ for $\ell'' \geq 2$, and so we have completed the proof of (4.7.10). Otherwise, by (4.7.12),

$$\begin{aligned}
&(4.7.14)\\
=&(CT^\epsilon)^{\ell+1+\ell'} 2^{m_j-1} \int_{[0,T)^{m_j-\ell-2-\ell'}} \|U_{\ell+3+\ell'} f^2_{\ell+3+\ell'}\|_{H^{s_\epsilon}} \cdots \|U_{\ell+3+\ell'} f^{2\ell+2\ell'+4}_{\ell+3+\ell'}\|_{H^{s_\epsilon}}\\
&\times \|(U_{\ell+3+\ell'} f^1_{\ell+3+\ell'})(U_{\ell+3+\ell'} f^2_{\ell+3+\ell'})(U_{\ell+3+\ell'} f^3_{\ell+3+\ell'})\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} dt_{\ell+2+\ell'} \cdots dt_{m_j-1}
\end{aligned}$$

which is of the form (4.7.11).

Case 3: $f^4_{\ell+2}$ is cubic. This case can be treated like Case 2. We choose $\ell' \geq 1$ satisfying (4.7.12), and combine linear propagators acting on $f^1_{\ell+2+\ell'}$. Then, we repeat the above procedure to bound (4.7.11) by (4.7.13). □

Next, we consider the contribution of the regular factors J^1_j.

Proposition 4.9. *Let $d \geq 3$. Then, for the regular tree τ_j, we have the bound*

$$\begin{aligned}
\int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \mathrm{Tr}\left(\left| S^{(1,-d)} J^1_j(t, t_1, \cdots, t_{m_j}; \sigma_j) \right| \right)\\
\leq 2^{m_j} C^{m_j} T^{\epsilon m_j} \|\phi\|^{2m_j+2}_{H^{s_\epsilon}}.
\end{aligned} \tag{4.7.16}$$

Similarly, when $d = 2$, we have the bound

$$\int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \operatorname{Tr}\left(\left| S^{(1,-d)} J_j^1(t, t_1, \cdots, t_{m_j}; \sigma_j) \right| \right) \\ \leq 2^{m_j} C^{m_j} T^{\frac{1}{3} m_j} \|\phi\|_{H^{1/3}}^{2m_j+2}, \tag{4.7.17}$$

and, when $d = 1$, we have the bound

$$\int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \operatorname{Tr}\left(\left| S^{(1,-d)} J_j^1(t, t_1, \cdots, t_{m_j}; \sigma_j) \right| \right) \\ \leq 2^{m_j} C^{m_j} T^{\frac{1}{2} m_j} \|\phi\|_{L^2}^{2m_j+2}. \tag{4.7.18}$$

Proof. Again, we consider the case $d \geq 3$, and note that the proof for $d = 1, 2$ is analogous (based on using the bounds for $d = 1, 2$ in Lemma 4.1).

We now proceed with the proof for $d \geq 3$.

$$\begin{aligned} &\int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \operatorname{Tr}\left(\left| S^{(1,-d)} J_j^1(t, t_1, \cdots, t_{m_j}; \sigma_j) \right| \right) \\ &= \int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \operatorname{Tr}\left(\left| S^{(1,-d)} U^{(1)}(t - t_1) \Theta_1 \right| \right) \\ &\leq \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \|\psi_{\beta_1}^1\|_{H^{-d}} \|\chi_{\beta_1}^1\|_{H^{-d}} \\ &\leq \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \|\psi_{\beta_1}^1\|_{H^{s_\epsilon}} \|\chi_{\beta_1}^1\|_{H^{s_\epsilon}} \end{aligned} \tag{4.7.19}$$

By (4.6.6) and (4.6.7), one of $\psi_{\beta_1}^1, \chi_{\beta_1}^1$ is cubic, and the other is linear. We define f_1^1 to be the cubic function, and f_1^2 to be the linear one. Then, by (4.6.6) and (4.6.7), f_1^1 and f_1^2 are of the form

$$f_1^1 = (U_2 f_2^1)(U_2 f_2^2)(U_2 f_2^3).$$

$$f_1^2 = U_2 f_2^4.$$

By (4.3.3), we have

$$(4.7.19) = \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \|(U_2 f_2^1)(U_2 f_2^2)(U_2 f_2^3)\|_{H^{s_\epsilon}} \|U_2 f_2^4\|_{H^{s_\epsilon}} \tag{4.7.20}$$

$$\leq (CT^\epsilon) \sum_{\beta_1} \int_{[0,T)^{m_j-1}} dt_2 \cdots dt_{m_j} \|f_2^1\|_{H^{s_\epsilon}} \|f_2^2\|_{H^{s_\epsilon}} \|f_2^3\|_{H^{s_\epsilon}} \|f_2^4\|_{H^{s_\epsilon}}. \tag{4.7.21}$$

By construction, only one of the factors f_2^ℓ is cubic. Without loss of generality, f_2^1 is cubic, and so we have

$$f_2^1 = (U_3 f_3^1)(U_3 f_3^2)(U_3 f_3^3),$$
$$f_2^\ell = U_3 f_3^{\ell+2} \quad \text{for } \ell = 2, 3, 4.$$

Thus,

$$(4.7.21) = (CT^\epsilon) \sum_{\beta_1} \int_{[0,T)^{m_j-1}} \|(U_3 f_3^1)(U_3 f_3^2)(U_3 f_3^3)\|_{H^{s_\epsilon}} \times \|U_3 f_3^4\|_{H^{s_\epsilon}} \|U_3 f_3^5\|_{H^{s_\epsilon}} \|U_3 f_3^6\|_{H^{s_\epsilon}} dt_2 \cdots dt_{m_j},$$

which is again of the form (4.7.20). Recall from §4.6.4 that there are at most 2^{m_j} terms in the sum over β_1. Repeating this argument $m_j - 1$ more times yields the desired result (4.7.16). $\square$

Before we proceed with the proof of Lemma 4.5, we present a short lemma that we use to bound the term $|\phi|^2\phi$ appearing on the right hand side of (4.7.1).

Lemma 4.10. *Let $\epsilon > 0$. Then, for $s_c = \frac{d}{2} - 1$, $r_\epsilon = \frac{2d}{d+2(1-\epsilon)}$, and $d \geq 3$, we have*

$$\||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \lesssim \|\phi\|^3_{H^{s_\epsilon}}. \tag{4.7.22}$$

Similarly, when $d = 2$, we have

$$\||\phi|^2\phi\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}),r_\epsilon}} \lesssim \|\phi\|^3_{H^{1/3}}. \tag{4.7.23}$$

Proof. Let $d \geq 3$. By two applications of the Sobolev inequality, we have

$$\||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \lesssim \||\phi|^2\phi\|_{L^{\frac{2d}{2d-\epsilon}}} = \|\phi\|^3_{L^{\frac{6d}{2d-\epsilon}}} \lesssim \|\phi\|^3_{H^{\frac{d+\epsilon}{6}}} \leq \|\phi\|^3_{H^{s_\epsilon}}.$$

This establishes (4.7.22).

Now let $d = 2$. In this case, the kernel G_α of the Bessel potential $< \nabla >^{-\alpha}$ behaves like $\frac{1}{|x|^{d-\alpha}}$ near 0 and decays exponentially like $e^{-c|x|}$ at ∞ (see Chapter V §3 of Stein's book [50]) for some $c > 0$. Thus

$$\begin{aligned}
\||\phi|^2\phi\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}),r_\epsilon}} &= \| < \nabla >^{-(\frac{1}{3}-\frac{\epsilon}{2})} (|\phi|^2\phi)\|_{L^{r_\epsilon}} = \| \int G_{\frac{1}{3}-\frac{\epsilon}{2}}(x-y)(|\phi|^2\phi)(y)dy\|_{L^{r_\epsilon}_x} \\
&\leq \int \|G_{\frac{1}{3}-\frac{\epsilon}{2}}(x-y)\|_{L^{r_\epsilon}_x}(|\phi|^3)(y)dy \qquad (4.7.24) \\
&\lesssim \int |\phi|^3 dy \qquad (4.7.25) \\
&\lesssim \|\phi\|^3_{H^{1/3}}.
\end{aligned}$$

To obtain (4.7.24) we use Minkowski integral inequality. (4.7.25) follows from the integrability of the Bessel potential, since when $d = 2$ and ϵ small, $(d - (\frac{1}{3} - \frac{\epsilon}{2}))r_\epsilon = (2 - \frac{1}{3} + \frac{\epsilon}{2})\frac{2}{2-\epsilon} < 2$. □

We are now ready to conclude the proof of Theorem 4.1 by proving Lemma 4.5.

Proof of Lemma 4.5. Recall from (4.6.1) that J^k can be decomposed into a product of k one particle kernels

$$J^k(t, t_1, \ldots, t_n; \sigma) = \prod_{j=1}^{k} J_j^1(t, t_{\ell_{j,1}}, \ldots, t_{\ell_{j,m_j}}; \sigma_j),$$

where only one of the factors J_j^1 distinguished. It now follows from Propositions 4.6 and 4.9 that

$$\begin{aligned}
&\int_{[0,T)^{n-1}} dt_1 \cdots dt_{n-1} \mathrm{Tr}\left(\left|S^{(k,-d)} J^k(t, t_1, \ldots, t_n; \sigma)\right|\right) \\
&= \int_{[0,T)^{n-1}} dt_1 \cdots dt_{n-1} \prod_{j=1}^{k} \mathrm{Tr}\left(\left|S^{(1,-d)} J_j^1(t, t_{\ell_{j,1}}, \ldots, t_{\ell_{j,m_j}}; \sigma_j)\right|\right) \\
&\leq \begin{cases} 2^n C^{n-1} T^{\epsilon(n-1)} \|\phi\|_{H^{s_\epsilon}}^{2(k+n)-3} \||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}), r_\epsilon}} & \text{if } d \geq 3 \\ 2^n C^{n-1} T^{\frac{1}{3}(n-1)} \|\phi\|_{H^{1/3}}^{2(k+n)-3} \||\phi|^2\phi\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}), r_\epsilon}} & \text{if } d = 2 \\ 2^n C^{n-1} T^{\frac{1}{2}(n-1)} \|\phi\|_{L^2}^{2(k+n)-3} \||\phi|^2\phi\|_{L^1} & \text{if } d = 1. \end{cases}
\end{aligned}$$

Thus, for $t \in [0, T)$, it follows from Lemma 4.10 that

$$\begin{aligned}
&\int_{[0,T)^{n-1}} d\underline{t}_{n-1} \mathrm{Tr}(|S^{(k,-d)} J^k(\underline{t}_n; \sigma)|) \\
&\leq \begin{cases} (CT^\epsilon)^{n-1} \|\phi\|_{H^{s_\epsilon}}^{2(k+n)} & \text{if } d \geq 3 \\ (CT^{1/3})^{n-1} \|\phi\|_{H^{1/3}}^{2(k+n)} & \text{if } d = 2 \\ (CT^{1/2})^{n-1} \|\phi\|_{H^{1/6}}^{2(k+n)} & \text{if } d = 1, \end{cases}
\end{aligned}$$

which is precisely the statement of Lemma 4.5. □

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